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Unique Reconstruction of Half-inverse Problems for Dirac System with Transmission Conditions Involving Spectral Parameter

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ABSTRACT

In this paper, the unique reconstruction for half-inverse problem of Dirac system with eigenvalue-dependent transmission conditions are investigated. The uniqueness theorem of the half-inverse problem is proved. Furthermore, an algorithm for reconstruction of the global solution of this nonlinear inverse problem is established.

General Terms

Inverse spectral problems, Dirac system

Keywords

Dirac system, Transmission conditions, Spectral parameter, Uniqueness theorem, Reconstruction method

1. INTRODUCTION

Boundary value problems with discontinuities inside the interval appear in mathematics, mechanics, physics, geophysics, radio electronics and other fields of natural properties [7]. As a ruler, such problems are related to discontinuous material properties (see [1, 11, 14] and references therein). The problem of reconstructing the material properties of a medium from data collected outside of the medium is of central importance in disciplines ranging from engineering to geosciences, where the data is general connected with the spectral information for the problems [7]. For example, discontinuous inverse problems appear in electronics for constructing parameters of heterogeneous electronic lines with desirable technical characteristics [18]. After reducing the corresponding mathematical model we come to the boundary value problem of differential equations [7]. Usually, the eigenvalue parameter appears linearly only in the differential equation. However, such problems are encountered in mathematical physics, which contain eigenvalue parameters not only in the differential equation, but also in the boundary and discontinuity conditions. These types of studies introduce qualitative changes in the exploration.

Dirac systems take an important role in both fundamental particle physics as well as condensed matter physics with applications in electronics and computation [21, 29]. They merge successfully quantum mechanics with special relativity, provide a natural description of the electron spin and predicting the existence of antimatter [1]. Furthermore, they are able to reproduce accurately the spectrum of the hydrogen atom and its realm, relativistic quantum mechanics, are considered as the natural transition to quantum field theory [9]. This paper deals with the system of Dirac differential equations

$$By' + Q(x)y = \lambda y \tag{1}$$

with

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, Q(x) = \begin{pmatrix} p(x) & r(x) \\ r(x) & -p(x) \end{pmatrix}, y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix},$$

where $p(x)$,$r(x)$ are complex-valued functions and they are absolutely continuous and λ is spectral parameter. We denote the boundary-value problem generated by (1) subject to the boundary conditions

$$\begin{aligned} &y_1(0)\cos\alpha + y_2(0)\sin\alpha = 0, \\ &y_1(\pi)\cos\beta + y_2(\pi)\sin\beta = 0, \end{aligned} \tag{2}$$

and the transmission conditions

$$\begin{aligned} &y_1(\frac{\pi}{2}+0) = h_1 y_1(\frac{\pi}{2}-0), \\ &y_2(\frac{\pi}{2}+0) = h_1^{-1} y_2(\frac{\pi}{2}-0) + (\lambda + h_2) y_1(\frac{\pi}{2}-0), \end{aligned} \tag{3}$$

where $(\alpha, \beta) \in [0, \pi) \times (0, \pi]$ and h_1,h_2 are complex numbers.

The basic and comprehensive inverse spectral result about Dirac systems is that the information of two spectra determines the potential pair of an Dirac equation uniquely [8], which is called the Borg's type theorem. The inverse spectral problems with mixed data have been studied for a number of years and a series of conclusions have been made (see for example [3, 5, 17, 19, 22, 23, 30] and the references therein). In particular, Amour [2] proved the half-inverse theorem, which says that the Dirichlet spectrum determines the potential pair (p, r) uniquely provided the potential pair (p, r) given a priori on the interval $[0, \pi/2]$ (or $[\pi/2, \pi]$). The solvability and numerical methods for half-inverse problem has been investigated by Yang and Liu [26] for the classical system and by Yang, Yurko, and Zhang for the system with transmission conditions [27].

Detailed studies on direct and inverse spectral problems for the Dirac systems with eigenvalue dependent boundary or transmission conditions have been studied fairly completely, see e.g. [6, 10, 12, 13, 24, 25, 28]. To the best of our knowledge, there is not much done in the literature on the reconstruction problem of inverse problems for these Dirac systems, contrasting to the uniqueness problem. When potential functions are real-valued and the boundary and transmission conditions constants are real numbers with $h_1 > 0$, in this case, the system is selfadjoint, i.e, all of the eigenvalues are real and simple, the authors [28] considered the uniqueness problem, they proved the Borg's type theorem and half-inverse theorem. In [25], when potential functions are complex-valued and the boundary and transmission conditions constants are complex numbers, that is, the system is non-selfadjoint in $L^2[0,\pi]\times L^2[0,\pi]$, the uniqueness theorems of the inverse problem according to the Weyl functions, two spectra, and one spectrum and according norming constant are proved. Furthermore, the constructive procedures for solving the Borg's type theorems are given. However, the half-inverse problem remains open. In [13, 25, 28], both the boundary conditions, and the transmission conditions depend on the spectral parameter, but in [6, 10, 24], only the boundary conditions depend on the spectral parameter. In this paper, we will consider the unique reconstruction of half-inverse problem for (1)-(3), where only transmission conditions depend on the spectral parameter.

Let $\sigma = \{\lambda_n\}_{n\in\mathbb{Z}}$ (counting with multiplicities) be the eigenvalues of the system (1)-(3), denoted by $\mathcal{H}$, and we define the input data set by

$$\Omega = \{\alpha, h_1, h_2, (p(x), r(x)) \text{ on } [0, \frac{\pi}{2}], \sigma\}. \tag{4}$$

We shall consider the following inverse problem:

problem 1 Given the data set Ω defined by (4), whether the Dirac system $\mathcal{H}$ is uniquely determined? How to reconstruct $(p(x), r(x))$ on the interval $(\pi/2, \pi]$ and β?

More precisely, we will prove the uniqueness theorem for Inverse problem 1, and suggest an algorithm for constructing the global solution of this nonlinear inverse problem.

Here is a sketch of the contents of this paper. In Section 2, we obtain the spectral characterization of the Dirac system $\mathcal{H}$ and provide the Lagrange interpolation theorem we will use. In Section 3, we prove the uniqueness theorem of Problem 1. In Section 4, basing on the uniqueness theorem, we reconstruct potential pair $(p(x), r(x))$ and β by using the input data Ω.

2. PRELIMINARIES

In this section, we derive some formulation of the inverse problem for $\mathcal{H}$. Throughout this paper, we always denote by $\mathcal{L}_a$ the class of entire functions of exponential type $\leq a$ which belong to $L^2(-\infty,+\infty)$ for real λ.

Let us consider the class of sine-type function introduced in [31]:

Definition A An entire function $f(\lambda)$ of exponential type $a > 0$ is said to be a sin-type function if

(1) the zeros of $f(\lambda)$ are separated, and

(2) there exist positive constants A, B, and H such that

$$Ae^{a|y|} \leq |f(x+iy)| \leq Be^{a|y|}$$

whenever x and y are real and $|y| \geq H$.

The following interpolation theorem (see, for example, [16, Theorem A]) and [15, Theorem 1, Lecture 21]) is corresponding to sine type functions, which plays an important role in our paper.

Theorem A (Lagrange Interpolation Theorem). Let $F(z)$ be a sine-type function with indicator diagram of width 2σ , and $\{z_k\}_{k\in\mathbb{Z}}$ be its zero set. Then the mapping

$$\{c_k\}_{k\in\mathbb{Z}} \mapsto f(z) = F(z)\sum_{k=-\infty}^{+\infty}\frac{c_k}{F'(z_k)(z-z_k)} \tag{5}$$

is an isomorphism between l^2 and $\mathcal{L}_\sigma$. The series on the right-hand side of (5) converges in the $L^2(R)$-norm. The inverse mapping is defined by the relation

$$f \mapsto \{f(z_k)\}_{k\in\mathbb{Z}}.$$

Let $\Phi(x,\lambda) = (\varphi_1(x,\lambda), \varphi_2(x,\lambda))^T$ and $\Psi(x,\lambda) = (\psi_1(x,\lambda), \psi_2(x,\lambda))^T$ be the solutions of (1) that satisfy the following initial conditions:

$$\Phi(0,\lambda) = (\sin\alpha, -\cos\alpha)^T, \tag{6}$$

$$\Psi(\pi,\lambda) = (\sin\beta, -\cos\beta)^T. \tag{7}$$

The characteristic function of the Dirac system $\mathcal{H}$ is denoted by

$$\Delta(\lambda) = \langle\Phi(x,\lambda), \Psi(x,\lambda)\rangle. \tag{8}$$

Note that $\Delta(\lambda)$ does not depend on x. Thus we get

$$\begin{aligned}\Delta(\lambda) &= \begin{vmatrix}\varphi_1\left(\frac{\pi}{2}+0,\lambda\right) & \psi_1\left(\frac{\pi}{2}+0,\lambda\right)\\ \varphi_2\left(\frac{\pi}{2}+0,\lambda\right) & \psi_2\left(\frac{\pi}{2}+0,\lambda\right)\end{vmatrix}\\ &= \begin{vmatrix}h_1\varphi_1\left(\frac{\pi}{2}-0,\lambda\right) & \psi_1\left(\frac{\pi}{2}+0,\lambda\right)\\ h_1^{-1}\varphi_2\left(\frac{\pi}{2}-0,\lambda\right) & \\ +(\lambda+h_2)\varphi_1\left(\frac{\pi}{2}-0,\lambda\right) & \psi_2\left(\frac{\pi}{2}+0,\lambda\right)\end{vmatrix}.\end{aligned} \tag{9}$$

Let us mention that if $(p(x), r(x))$ is known on $(0,\pi/2)$, then $\varphi_1\left(\frac{\pi}{2}-0,\lambda\right), \varphi_2\left(\frac{\pi}{2}-0,\lambda\right)$ are obtain. The main aim of this paper is, applying Theorem A, to unique reconstruct $\psi_1\left(\frac{\pi}{2}+0,\lambda\right)$ and $\psi_2\left(\frac{\pi}{2}+0,\lambda\right)$.

One knows that $\Phi(x,\lambda) = (\varphi_1(x,\lambda), \varphi_2(x,\lambda))^T$ and $\Psi(x,\lambda) = (\psi_1(x,\lambda), \psi_2(x,\lambda))^T$ are entire in λ, and the following asymptotics can be obtained [17, 20]:

$$\varphi_1(x,\lambda) = \begin{cases}\sin(\lambda x+\alpha) + \mathcal{A}_{1,-}(x,\lambda), & x < \frac{\pi}{2},\\ \frac{\lambda+h_2}{2}[\cos(\lambda x+\alpha) - \cos(\lambda(\pi-x)+\alpha)] & \\ \quad + \frac{h_1^2+1}{2h_1}\sin(\lambda x+\alpha) + \frac{h_1^2-1}{2h_1}\sin(\lambda(\pi & \\ \quad -x)+\alpha) + \mathcal{A}_{1,+}(x,\lambda), & x > \frac{\pi}{2},\end{cases} \tag{10}$$

$$\varphi_2(x,\lambda) = \begin{cases}-\cos(\lambda x+\alpha) + \mathcal{A}_{2,-}(x,\lambda), & x < \frac{\pi}{2},\\ \frac{\lambda+h_2}{2}[\sin(\lambda(\pi-x)+\alpha) + \sin(\lambda x+\alpha)] & \\ \quad - \frac{h_1^2+1}{2h_1}\cos(\lambda x+\alpha) + \frac{h_1^2-1}{2h_1}\cos(\lambda(\pi & \\ \quad -x)+\alpha) + \mathcal{A}_{2,+}(x,\lambda), & x > \frac{\pi}{2},\end{cases} \tag{11}$$

and

$$\psi_1(x,\lambda)=\begin{cases}\frac{\lambda+h_2}{2}[\cos(\lambda x-\beta)-\cos(\lambda(\pi-x)-\beta)]\\ \quad -\frac{h_1^2+1}{2h_1}\sin(\lambda(\pi-x)-\beta)\\ \quad +\frac{h_1^2-1}{2h_1}\sin(\lambda x-\beta)+\mathcal{B}_{1,-}(x,\lambda), & x<\frac{\pi}{2},\\ -\sin(\lambda(\pi-x)-\beta)+\mathcal{B}_{1,+}(x,\lambda), & x>\frac{\pi}{2},\end{cases} \tag{12}$$

$$\psi_2(x,\lambda)=\begin{cases}\frac{\lambda+h_2}{2}[\sin(\lambda(\pi-x)-\beta)+\sin(\lambda x-\beta)]\\ \quad -\frac{h_1^2+1}{2h_1}\cos(\lambda(\pi-x)-\beta)\\ \quad -\frac{h_1^2-1}{2h_1}\cos(\lambda x-\beta)+\mathcal{B}_{2,-}(x,\lambda), & x<\frac{\pi}{2},\\ -\cos(\lambda(\pi-x)-\beta)+\mathcal{B}_{2,+}(x,\lambda), & x>\frac{\pi}{2},\end{cases} \tag{13}$$

where $\mathcal{A}_{j,\pm}(x,\lambda)\in\mathcal{L}_x$ and $\mathcal{B}_{j,\pm}(x,\lambda)\in\mathcal{L}_{\pi-x}$ for $j=1,2$.

Combining with (3) and (9)-(13) we have

$$\begin{aligned}\Delta(\lambda)&=-(\lambda+h_2)\sin\left(\frac{\lambda\pi}{2}+\alpha\right)\sin\left(-\frac{\lambda\pi}{2}+\beta\right)\\&\quad+h_1^{-1}\cos\left(\frac{\lambda\pi}{2}+\alpha\right)\sin\left(-\frac{\lambda\pi}{2}+\beta\right)\\&\quad-h_1\sin\left(\frac{\lambda\pi}{2}+\alpha\right)\cos\left(-\frac{\lambda\pi}{2}+\beta\right)+\mathcal{A}(\lambda),\end{aligned} \tag{14}$$

where $\mathcal{A}(\lambda)\in\mathcal{L}_\pi$. Here the zeros of $\Delta(\lambda)$, denoted by $\sigma=\{\lambda_n\}_{n\in\mathbb{Z}}$, coincide with the eigenvalues of the system $\mathcal{H}$. By a well known method (see, for example, [7]), it is easily seen that the zeros of $\Delta(\lambda)$ are not all simple, the multiplicities of some zeros are greater than 1, and some roots are complex.

By Hadamard's factorization theorem [31, p.74] it is also known that $\Delta(\lambda)$ has the following representation:

$$\Delta(\lambda)=C\,(\lambda-\lambda_0)\prod_{n=1}^{\infty}\left(1-\frac{\lambda}{\lambda_n}\right)\left(1-\frac{\lambda}{\lambda_{-n}}\right). \tag{15}$$

Here, using (14), the constant C can be obtained by

$$\begin{aligned}C^{-1}&=\lim_{\tau\to+\infty}\frac{4}{\tau[e^{i(\alpha+\beta)}-e^{\tau\pi}e^{i(\beta-\alpha)}+e^{-i(\alpha+\beta)}]}\\&\quad\cdot\prod_{n=1}^{\infty}\left(1-\frac{i\tau}{\lambda_n}\right)\left(1-\frac{i\tau}{\lambda_{-n}}\right),\end{aligned} \tag{16}$$

where $i=\sqrt{-1}$.

3. THE UNIQUENESS PROBLEM

In this section, for Problem 1, we consider the uniqueness problem for the Dirac system $\mathcal{H}$, that is, we will prove that the potential functions pair $(p(x),r(x))$ and the boundary constant β are uniquely determined if the data set Ω is known a priori. More precisely, we will prove that the potential functions pair $(p(x),r(x))=(\tilde{p}(x),\tilde{r}(x))$ and the boundary constant $\beta=\tilde{\beta}$ by using Wely function if the data set $\Omega=\tilde{\Omega}$. Here relative to $\mathcal{H}$, the problem $\tilde{\mathcal{H}}$ is of the same form but with different potential function pair $(\tilde{p},\tilde{r})$ and different coefficients $\tilde{\alpha},\tilde{\beta},\tilde{h}_1,\tilde{h}_2$. We agree that if a certain symbol δ denotes an object related to $\mathcal{H}$, then $\tilde{\delta}$ will denote the analogous object related to $\tilde{\mathcal{H}}$.

Define the function $M(x,\lambda)=(m_1(x,\lambda),m_2(x,\lambda))^T$ by

$$M(x,\lambda)=\frac{\Psi(x,\lambda)}{\Delta(\lambda)}, \tag{17}$$

which is called the Wely solution of $\mathcal{H}$ [7]. Thus $M(x,\lambda)$ is meromorphic function and clearly it can also be given by

$$M(x,\lambda)=-\Theta(x,\lambda)+W(\lambda)\Phi(x,\lambda), \tag{18}$$

where $\Theta(x,\lambda)=(\theta_1(x,\lambda),\theta_2(x,\lambda))^T$ is the solution of (1) which satisfying the initial conditions $\Theta(0,\lambda)=(\cos\alpha,\sin\alpha)^T$ and

$$W(\lambda)=\frac{\psi_1(0,\lambda)\sin\alpha-\psi_2(0,\lambda)\cos\alpha}{\Delta(\lambda)} \tag{19}$$

is the Wely function of $\mathcal{H}$. Thus we have

$$\langle M(x,\lambda),\Phi(x,\lambda)\rangle=1. \tag{20}$$

Theorem 3.1 Let the set Ω be defined by (4). If $\Omega=\tilde{\Omega}$, then $(p(x),r(x))=(\tilde{p}(x),\tilde{r}(x))$ on $(\pi/2,\pi]$ and $\beta=\tilde{\beta}$.

PROOF. Write (1) in the following form of equations

$$\begin{cases}y_2'(x,\lambda)+p(x)y_1(x,\lambda)+r(x)y_2(x,\lambda)=\lambda y_1(x,\lambda),\\ -y_1'(x,\lambda)+r(x)y_1(x,\lambda)-p(x)y_2(x,\lambda)=\lambda y_2(x,\lambda).\end{cases} \tag{21}$$

Similarly we have

$$\begin{cases}\tilde{y}_2'(x,\lambda)+\tilde{p}(x)\tilde{y}_1(x,\lambda)+\tilde{r}(x)\tilde{y}_2(x,\lambda)=\lambda\tilde{y}_1(x,\lambda),\\ -\tilde{y}_1'(x,\lambda)+\tilde{r}(x)\tilde{y}_1(x,\lambda)-\tilde{p}(x)\tilde{y}_2(x,\lambda)=\lambda\tilde{y}_2(x,\lambda).\end{cases} \tag{22}$$

Now if we multiply the first equation of (21) by $\tilde{y}_1(x,\lambda)$, the second equation of (21) by $\tilde{y}_2(x,\lambda)$ and first and second equations of (22) by $y_1(x,\lambda),y_2(x,\lambda)$ respectively and subtract from each other, then integrating the above relation on the interval $[0,\pi]$ and using (6), we get

$$G(\lambda)=\psi_1(0,\lambda)\tilde{\psi}_2(0,\lambda)-\psi_2(0,\lambda)\tilde{\psi}_1(0,\lambda). \tag{23}$$

Here

$$\begin{aligned}G(\lambda)=\int_{\frac{\pi}{2}}^{\pi}&[(p(x)-\tilde{p}(x))(y_1(x,\lambda)\tilde{y}_1(x,\lambda)-y_2(x,\lambda)\tilde{y}_2(x,\lambda))\\&+(r(x)-\tilde{r}(x))(y_1(x,\lambda)\tilde{y}_2(x,\lambda)\\&+\tilde{y}_1(x,\lambda)y_2(x,\lambda))]dx+\sin(\beta-\tilde{\beta}).\end{aligned} \tag{24}$$

The function $G(\lambda)$ is an entire function since $\psi_i(x,\lambda)$ and $\tilde{\psi}_i(x,\lambda)$ are entire in λ. Substituting (12) and (13) into (23) it yields that

$$G(\lambda)=\sin(\tilde{\beta}-\beta)+\mathcal{S}(\lambda) \tag{25}$$

with $\mathcal{S}(\lambda)\in\mathcal{L}_{\pi/2}$.
Put

$$J(\lambda)=\frac{G(\lambda)}{\Delta(\lambda)}. \tag{26}$$

Next we shall show $J(\lambda)\equiv 0$. O prove that $J(\lambda)$ is an entire function. Since $\Omega=\{\lambda_n\}_{n\in\mathbb{Z}}=\tilde{\Omega}$ and $\alpha=\tilde{\alpha}$, we have from (1.2) that

$$\psi_1(0,\lambda_n)=-\tan\alpha\psi_2(0,\lambda_n),\ \ \tilde{\psi}_1(0,\lambda_n)=-\tan\alpha\tilde{\psi}_2(0,\lambda_n), \tag{27}$$

which together with (23) yields that $G(\lambda_n) = 0$. We note that $\Delta(\lambda_n) = 0$, therefore $J(\lambda)$ is an entire function. On the other hand, from (25) combining with (14) we have

$$J(\lambda) = O(\frac{1}{\lambda e^{|\mathrm{Im}\lambda|\pi}}).$$

This yields that

$$\lim_{\lambda\to\infty,\lambda\in\mathbb{R}} J(\lambda) = 0. \tag{28}$$

Furthermore one obtains $J(\lambda) \equiv 0$. T his combing with (26) implies that

$$G(\lambda) \equiv 0. \tag{29}$$

Thus in virtue of (25) we have

$$\beta = \tilde{\beta}. \tag{30}$$

Moreover, (29) together with (23)-(24) and (12)-(13) yields that

$$\psi_1(0,\lambda) = \tilde{\psi}_1(0,\lambda) \text{ and } \psi_2(0,\lambda) = \tilde{\psi}_2(0,\lambda). \tag{31}$$

Substituting (30) into (16) we have $C = \tilde{C}$, i.e.,

$$\Delta(\lambda) = \tilde{\Delta}(\lambda). \tag{32}$$

In virtue of (19), (31), and (32) we obtain

$$W(\lambda) = \tilde{W}(\lambda). \tag{33}$$

Let us define a matrix $P(x,\lambda) = [p_{i,j}(x,\lambda)]_{i,j=1,2}$ by the formula

$$P(x,\lambda)\begin{bmatrix} \tilde{\varphi}_1(x,\lambda) & \tilde{m}_1(x,\lambda) \\ \tilde{\varphi}_2(x,\lambda) & \tilde{m}_2(x,\lambda) \end{bmatrix} = \begin{bmatrix} \varphi_1(x,\lambda) & m_1(x,\lambda) \\ \varphi_2(x,\lambda) & m_2(x,\lambda) \end{bmatrix}. \tag{34}$$

Using (34) we get

$$\begin{aligned} \varphi_1(x,\lambda) &= p_{11}(x,\lambda)\tilde{\varphi}_1(x,\lambda) + p_{12}(x,\lambda)\tilde{\varphi}_2(x,\lambda), \\ \varphi_2(x,\lambda) &= p_{21}(x,\lambda)\tilde{\varphi}_1(x,\lambda) + p_{22}(x,\lambda)\tilde{\varphi}_2(x,\lambda), \\ m_1(x,\lambda) &= p_{11}(x,\lambda)\tilde{m}_1(x,\lambda) + p_{12}(x,\lambda)\tilde{m}_2(x,\lambda), \\ m_2(x,\lambda) &= p_{21}(x,\lambda)\tilde{m}_1(x,\lambda) + p_{22}(x,\lambda)\tilde{m}_2(x,\lambda). \end{aligned} \tag{35}$$

From (20) one can convert (35) into the following form

$$\begin{aligned} p_{11}(x,\lambda) &= \tilde{\varphi}_2(x,\lambda)m_1(x,\lambda) - \varphi_1(x,\lambda)\tilde{m}_2(x,\lambda), \\ p_{12}(x,\lambda) &= \varphi_1(x,\lambda)\tilde{m}_1(x,\lambda) - \tilde{\varphi}_1(x,\lambda)m_1(x,\lambda), \\ p_{21}(x,\lambda) &= \tilde{\varphi}_2(x,\lambda)m_2(x,\lambda) - \varphi_2(x,\lambda)\tilde{m}_2(x,\lambda), \\ p_{22}(x,\lambda) &= \varphi_2(x,\lambda)\tilde{m}_1(x,\lambda) - \tilde{\varphi}_1(x,\lambda)m_2(x,\lambda). \end{aligned} \tag{36}$$

Substituting (18) and (33) into (36) one obtains

$$\begin{aligned} p_{11}(x,\lambda) &= \varphi_1(x,\lambda)\tilde{\theta}_2(x,\lambda) - \tilde{\varphi}_2(x,\lambda)\theta_1(x,\lambda), \\ p_{12}(x,\lambda) &= \tilde{\varphi}_1(x,\lambda)\theta_1(x,\lambda) - \varphi_1(x,\lambda)\tilde{\theta}_1(x,\lambda), \\ p_{21}(x,\lambda) &= \varphi_2(x,\lambda)\tilde{\theta}_2(x,\lambda) - \tilde{\varphi}_2(x,\lambda)\theta_2(x,\lambda), \\ p_{22}(x,\lambda) &= \tilde{\varphi}_1(x,\lambda)\theta_2(x,\lambda) - \varphi_2(x,\lambda)\tilde{\theta}_1(x,\lambda). \end{aligned} \tag{37}$$

So

$$\begin{aligned} p_{11}(x,\lambda) - 1 &= \frac{\tilde{\psi}_2(x,\lambda)[\tilde{\varphi}_1(x,\lambda) - \varphi_1(x,\lambda)]}{\tilde{\Delta}(\lambda)} \\ &\quad + \tilde{\varphi}_2(x,\lambda)\left[\frac{\psi_1(x,\lambda)}{\Delta(\lambda)} - \frac{\tilde{\psi}_1(x,\lambda)}{\tilde{\Delta}(\lambda)}\right], \\ p_{12}(x,\lambda) &= \frac{\psi_1(x,\lambda)[\varphi_1(x,\lambda) - \tilde{\varphi}_1(x,\lambda)]}{\Delta(\lambda)} \\ &\quad + \varphi_1(x,\lambda)\left[\frac{\tilde{\psi}_1(x,\lambda)}{\tilde{\Delta}(\lambda)} - \frac{\psi_1(x,\lambda)}{\Delta(\lambda)}\right], \\ p_{21}(x,\lambda) &= \frac{\tilde{\psi}_2(x,\lambda)[\tilde{\varphi}_2(x,\lambda) - \varphi_2(x,\lambda)]}{\tilde{\Delta}(\lambda)} \\ &\quad + \tilde{\varphi}_2(x,\lambda)\left[\frac{\psi_2(x,\lambda)}{\Delta(\lambda)} - \frac{\tilde{\psi}_2(x,\lambda)}{\tilde{\Delta}(\lambda)}\right], \\ p_{22}(x,\lambda) - 1 &= \frac{\psi_2(x,\lambda)[\varphi_1(x,\lambda) - \tilde{\varphi}_1(x,\lambda)]}{\Delta(\lambda)} \\ &\quad + \varphi_2(x,\lambda)\left[\frac{\tilde{\psi}_1(x,\lambda)}{\tilde{\Delta}(\lambda)} - \frac{\psi_1(x,\lambda)}{\Delta(\lambda)}\right]. \end{aligned} \tag{38}$$

From (10)-(13) we get

$$\lim_{\lambda\to\infty,\lambda\in\mathbb{R}} \frac{\tilde{\psi}_2(x,\lambda)[\tilde{\varphi}_1(x,\lambda) - \varphi_1(x,\lambda)]}{\tilde{\Delta}(\lambda)} = 0$$

and

$$\lim_{\lambda\to\infty,\lambda\in\mathbb{R}} \tilde{\varphi}_2(x,\lambda)\left[\frac{\psi_1(x,\lambda)}{\Delta(\lambda)} - \frac{\tilde{\psi}_1(x,\lambda)}{\tilde{\Delta}(\lambda)}\right] = 0.$$

For all $x \in (0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi)$, therefore we have from (38) that

$$\lim_{\lambda\to\infty,\lambda\in\mathbb{R}} [p_{11}(x,\lambda) - 1] = 0.$$

Then we get $p_{11}(x,\lambda) = 1$ since $p_{11}(x,\lambda)$ is entire function. Similarly we obtain $p_{12}(x,\lambda) = 0$, $p_{21}(x,\lambda) = 0$, and $p_{22}(x,\lambda) = 1$. Now (35) and this result give $\Phi(x,\lambda) = \tilde{\Phi}(x,\lambda)$ and $M(x,\lambda) = \tilde{M}(x,\lambda)$ for all $x \in (0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi)$. Clearly by (18) and (32) we know that $\Psi(x,\lambda) = \tilde{\Psi}(x,\lambda)$. So $(p(x), r(x)) = (\tilde{p}(x), \tilde{r}(x))$ on $(\frac{\pi}{2}, \pi]$. This completes the proof. □

4. THE RECONSTRUCTION METHOD

In this section we shall solve Problem 1, that is, provide the method for recovering β and the potential pair (p, r) on $(\pi/2, \pi]$ in terms of the spectral data set Ω by using Theorem A.

Let $c_k = \{2k - \frac{2\alpha}{\pi}\}_{k\in\mathbb{Z}}$. It is evident that

$$\sin(\frac{c_k\pi}{2} + \alpha) = 0 \qquad \text{and} \qquad \cos(\frac{c_k\pi}{2} + \alpha) = (-1)^k. \tag{39}$$

Substituting (39) into (14) together with (15) one gets that

$$C^{-1} = h_1 \csc(\alpha+\beta) \lim_{k\to+\infty} (c_k - \lambda_0) \prod_{n=1}^{\infty} \left(1 - \frac{c_k}{\lambda_n}\right)\left(1 - \frac{c_k}{\lambda_{-n}}\right). \tag{40}$$

Note that $\varphi_1\left(\frac{\pi}{2}-0,\lambda\right)$ and $\varphi_2\left(\frac{\pi}{2}-0,\lambda\right)$ are sin-type functions. In virtue of (3), $\Delta(\lambda)$ can be written as

$$\Delta(\lambda)=\begin{vmatrix} h_1\varphi_1\left(\frac{\pi}{2}-0,\lambda\right) & \psi_1\left(\frac{\pi}{2}+0,\lambda\right) \\ h_1^{-1}\varphi_2\left(\frac{\pi}{2}-0,\lambda\right) & \\ +(\lambda+h_2)\varphi_1\left(\frac{\pi}{2}-0,\lambda\right) & \psi_2\left(\frac{\pi}{2}+0,\lambda\right)\end{vmatrix}. \quad (41)$$

If potential pair $(p(x),r(x))$ on $(0,\pi/2)$ and α are given, then we can find $\varphi_1\left(\frac{\pi}{2}-0,\lambda\right)$ and its zeros $\{v_n\}_{n\in\mathbb{Z}}$ as well as $\varphi_2\left(\frac{\pi}{2}-0,\lambda\right)$ and its zeros $\{u_n\}_{n\in\mathbb{Z}}$. Moreover, it is easy to know that the two sequences have the following asymptotics representations:

$$v_n=2n-\frac{2\alpha}{\pi}+\beta_n, \quad (42)$$

$$u_n=2n+1-\frac{2\alpha}{\pi}+\varepsilon_n, \quad (43)$$

where the sequences $\{\beta_n\}_{n\in\mathbb{Z}},\{\varepsilon_n\}_{n\in\mathbb{Z}}\in l^2$. Substituting $\lambda=v_n$ into (41), we get

$$\psi_1\left(\frac{\pi}{2}-0,v_n\right)=-\frac{\Delta(v_n)}{\varphi_2\left(\frac{\pi}{2}-0,v_n\right)}.$$

This together with (12) yields that

$$\begin{aligned}\mathcal{B}_{1,-}(\frac{\pi}{2}-0,v_n)&=-\frac{\Delta(v_n)}{\varphi_2\left(\frac{\pi}{2}-0,v_n\right)}+h_1^{-1}\sin(\frac{v_n\pi}{2}-\beta)\\ &=:\vartheta_n \end{aligned}\quad (44)$$

Note that from (14), (11), and (42) we have

$$\begin{aligned}\Delta(v_n)&=h_1^{-1}\sin(\alpha+\beta)+\gamma_n,\\ \varphi_2\left(\frac{\pi}{2}-0,v_n\right)&=(-1)^{n+1}+\xi_n,\\ \sin(\frac{v_n\pi}{2}-\beta)&=(-1)^{n+1}\sin(\alpha+\beta)+\zeta_n,\end{aligned}$$

where $\{\gamma_n\}_{n\in\mathbb{Z}},\{\xi_n\}_{n\in\mathbb{Z}}$, and $\{\zeta_n\}_{n\in\mathbb{Z}}\in l^2$. Substituting the above results into (44) and by simple calculation we get $\{\vartheta_n\}_{n\in\mathbb{Z}}\in l^2$. Thus applying Theorem A we get

$$\mathcal{B}_{1,-}(\tfrac{\pi}{2}-0,\lambda)=\varphi_1(\tfrac{\pi}{2}-0,\lambda)\sum_{n=-\infty}^{+\infty}\frac{\vartheta_n}{\frac{d\varphi_1(\frac{\pi}{2}-0,\lambda)}{d\lambda}|_{\lambda=v_n}(\lambda-v_n)}. \quad (45)$$

The series on the right-hand side of (45) converges uniformly to a function which belongs to $\mathcal{L}_{\frac{\pi}{2}}$. Then $\psi_1(\frac{\pi}{2}-0,\lambda)$ can be constructed from (12).

Moreover, substituting $\lambda=u_n$ into equation (41) we get

$$\psi_2\left(\tfrac{\pi}{2}-0,u_n\right)=\frac{\Delta(u_n)}{\varphi_1\left(\frac{\pi}{2}-0,u_n\right)},$$

which in combination with (13) yields that

$$\begin{aligned}\mathcal{B}_{2,-}(\frac{\pi}{2}-0,u_n)&=\frac{\Delta(u_n)}{\varphi_1\left(\frac{\pi}{2}-0,u_n\right)}-(u_n+h_2)\sin(\frac{u_n\pi}{2}-\beta)\\ &+h_1\cos(\frac{u_n\pi}{2}-\beta)\\ &=:\hat{\vartheta}_n\end{aligned}\quad (46)$$

From (14), (10), and (43) we have

$$\begin{aligned}\Delta(u_n)&=(1+2n-\frac{2\alpha}{\pi}+h_2)\cos(\alpha+\beta)\\ &-h_1\sin(\alpha+\beta)+\hat{\gamma}_n,\\ \varphi_1\left(\frac{\pi}{2}-0,u_n\right)&=(-1)^n+\hat{\xi}_n,\\ \sin(\frac{u_n\pi}{2}-\beta)&=(-1)^n\cos(\alpha+\beta)+\hat{\zeta}_n,\\ \cos(\frac{u_n\pi}{2}-\beta)&=(-1)^n\sin(\alpha+\beta)+\hat{\kappa}_n,\end{aligned}$$

where $\{\hat{\gamma}_n\}_{n\in\mathbb{Z}},\{\hat{\xi}_n\}_{n\in\mathbb{Z}},\{\hat{\zeta}_n\}_{n\in\mathbb{Z}}$, and $\{\hat{\kappa}_n\}_{n\in\mathbb{Z}}\in l^2$. Substituting the above results into (46) and by simple calculation we get $\{\hat{\vartheta}_n\}_{n\in\mathbb{Z}}\in l^2$. So applying Theorem A we get

$$\mathcal{B}_{2,-}(\tfrac{\pi}{2}-0,\lambda)=\varphi_2(\tfrac{\pi}{2}-0,\lambda)\sum_{n=-\infty}^{+\infty}\frac{\hat{\vartheta}_n}{\frac{d\varphi_2(\frac{\pi}{2}-0,\lambda)}{d\lambda}|_{\lambda=u_n}(\lambda-u_n)}. \quad (47)$$

Then $\psi_2(\frac{\pi}{2}-0,\lambda)$ can be reconstructed from (13).

Based on the above discussion, the algorithm for solving Problem 1 is shown below.

Algorithm for Problem 1: Let the input data set Ω be given.

(1) Find β by solving (40) together with (16).
(2) Construct $\Delta(\lambda)$ via (15).
(3) Compute the solutions $\varphi_1(x,\lambda)$ and $\varphi_2(x,\lambda)$ of (1) for $x\in(0,\pi/2)$, then get the zeros $\{v_n\}_{n\in\mathbb{Z}}$ and $\{u_n\}_{n\in\mathbb{Z}}$ of $\varphi_1\left(\frac{\pi}{2}-0,\lambda\right)$ and $\varphi_2\left(\frac{\pi}{2}-0,\lambda\right)$ respectively.
(4) Construct $\mathcal{B}_{1,-}(\frac{\pi}{2}-0,\lambda)$ via (44) together with (45) and then obtain $\psi_1(\frac{\pi}{2}-0,\lambda)$ by (12); Construct $\mathcal{B}_{2,-}(\frac{\pi}{2}-0,\lambda)$ via (46) together with (47) and then obtain $\psi_2(\frac{\pi}{2}-0,\lambda)$ by (13).
(5) Obtain $\psi_1(\frac{\pi}{2}+0,\lambda)$ and $\psi_2(\frac{\pi}{2}+0,\lambda)$ in virtue of (3).
(6) Construct the function pair (p,r) on $(\pi/2,\pi]$ by the procedure described in [4, Section 2].

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