International Journal of Computer Applications (0975 - 8887)
Volume 187 - No.68, December 2025

Independent Transversal Geodetic Number of a Graph

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ABSTRACT

A subset $S \subseteq V$ is said to be a *geodetic set* in G = (V, E) if each vertex of G lies on at least one shortest path between some pair of vertices $u, v \in S$. The cardinality of the minimum geodetic set is known as *geodetic number* of G, denoted by $g(G)$ [4, 5]. A subset $I \subseteq V$ is said to be independent if there is no edge between every pair of vertices $u, v \in I$ [3]. A geodetic set $S \subseteq V(G)$ that intersects every maximum independent set (β_0-set) of G is called an independent transversal geodetic set. This study produces the new notion of *independent transversal geodetic number* among geodetic sets. It explores the characteristics of this new parameter within several well-known graph families and offer an in-depth investigation of the fundamental properties of *independent transversal geodetic number*.

General Terms

Geodetic number, Independent transversal geodetic number

Keywords

Geodetic set, Independent transversal geodetic set, Independent transversal geodetic number

1. INTRODUCTION

Throughout this study, $G = (V, E)$ is a simple, undirected and connected graph. The order of G is referred to as $|V| = n$, a finite positive number [1, 3].
The distance between two vertices u and v in G, denoted by $d(u, v)$ is defined as the length of a shortest path connecting them in G [1].

The eccentricity of a vertex v, denoted by $e(v)$ is defined as the distance between v and a vertex farthest from v in G. The minimum value among the eccentricity of each vertex in G is called as the radius of G, denoted by $rad(G)$, whereas the maximum value is called as the diameter of G, denoted by $diam(G)$.
If the sub graph induced by the neighbours of a vertex v forms a clique in G, then v is said to be an extreme vertex. All these concepts are referred in [2].
A set $S \subseteq V(G)$ is called a *geodetic set* if every vertex of G lies on a shortest u-v path between some pair of vertices $u, v \in S$. The minimum cardinality among all geodetic sets is called *geodetic number* and is denoted by $g(G)$. Any geodetic set of minimum cardinality is referred to as g-set. The geodetic number of a graph is studied in [4, 5].
A subset $I \subseteq V$ is said to be independent if there is no edge between every pair of vertices $u, v \in I$ in G [3]. A independent set I is maximum only when $|I| > |J|$ for every other independent set J in the graph [3]. Maximum independent sets are referred to as β_0-sets [8, 9].
The concepts of Independent transversal domination number and Independent transversal Steiner number have been introduced and studied in [6, 10].
This study proposes the notion of *independent transversal geodetic number* as a novel graph invariant.
The theorems listed below are referred where appropriate.

THEOREM 1.1. *[4] Each extreme vertex (end vertex) of a graph G belongs to every geodetic set of G.*

THEOREM 1.2. *[4] For a connected graph G, $g(G) = n$ iff $G = K_n$.*

THEOREM 1.3. *[4] The geodetic number of a tree T is equal to the number of end vertices in T.*

THEOREM 1.4. *[7] If $G = K_n - \{e\}$ where e is any egde of K_n, then $g(G) = 2$.*

2. DEFINITIONS AND EXAMPLES

DEFINITION 2.1. *Let $G = (V, E)$ be a simple connected graph having three or more vertices. A geodetic set $S \subseteq V(G)$ that intersects every maximum independent set (β_0-set) of G is called an independent transversal geodetic set.*

DEFINITION 2.2. *The minimum cardinality of an independent transversal geodetic set is called the independent transversal geodetic number of G, denoted by $g_{it}(G)$.*

REMARK 2.3. *An independent transversal geodetic set of minimum cardinality is referred to as g_{it}-set.*

EXAMPLE 1. *The sets $\{x_1, x_3\}$, $\{x_1, x_4\}$, $\{x_1, x_5\}$, $\{x_2, x_4\}$, $\{x_2, x_5\}$, $\{x_3, x_6\}$ and $\{x_4, x_6\}$ are β_0- sets in the graph given in Figure 1 [10]*

.
$S = \{x_1, x_2, x_4, x_6\}$ is a geodetic set which intersects every β_0-set in G.
So S is a g_{it}-set and $g_{it}(G) = 4$.

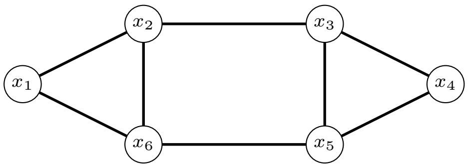

Fig. 1. Graph G

OBSERVATION 2.4. *For the graph G depicted in Figure 1 of Example 1, the minimum geodetic set is unique and is composed of the vertices* x_1 *and* x_4*, which establishes* $g(G) = 2$*. The observation is then made that this graph provides an instance where* $g(G) \neq g_{it}(G)$*.*

EXAMPLE 2. *The sets* $I_1 = \{x_2, x_4, x_6\}$ *and* $I_2 = \{x_2, x_5, x_6\}$ *are the only* β_0*-sets for the graph G in Figure 2.*

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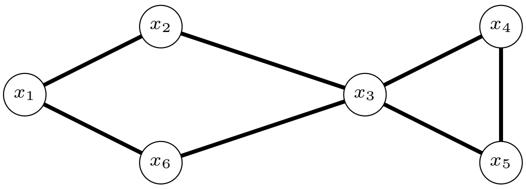

Fig. 2. Graph G

The set $S = \{x_1, x_4, x_5\}$ *is a geodetic set that intersects both* I_1 *and* I_2*.*
$\therefore$ *S is a* g_{it}*-set and so* $g_{it}(G) = 3$*.*

REMARK 2.5. *In Example 2, the geodetic number* $g(G)$ *is 3, with* $\{x_1, x_4, x_5\}$ *being the unique minimum geodetic set. This set also satisfies the conditions for an independent transversal geodetic set, demonstrating that the two numbers are equal.*

3. g_{it} OF SOME FAMILIAR GRAPHS

RESULT 3.1. $g_{it}(K_n) = n$ *since* $\{x_i\}$*,* $i = 1, 2, ..., n$ *are the* β_0*-sets in* K_n *and the set* $S = \{x_1, x_2, ..., x_n\}$ *is the unique geodetic set that intersects every* β_0*-set in* K_n*.*

RESULT 3.2. *Let* $G = K_n - \{e\}$ *where* $n \geq 3$ *and* $e = x_i x_j$ *is any edge of* K_n *for* $i, j = 1, 2, ..., n$ *with* $i \neq j$ *.*
Here $I = \{x_i, x_j\}$ *is the unique* β_0*-set in G which is also a geodetic set of minimum cardinality in G.*
So $g_{it}(K_n - \{e\}) = 2$*.*

REMARK 3.3. *According to Theorems 1.2 and 1.4, the geodetic number matches the independent transversal geodetic number for both* K_n *and* $K_n - \{e\}$*. It is observed that when a single edge is deleted from* K_n *for* $n \geq 3$*, both parameters are substantially reduced.*

RESULT 3.4. *Let G be a star graph with* $n+1$ *vertices labeled* u*,* x_1*,* x_2*, ...,* x_n *with* u *as the central vertex. Then* $g_{it}(G) = n$ *since* $\{x_1, x_2, ..., x_n\}$ *is the unique minimum geodetic set, which is also the unique* β_0*-set in G.*

NOTE 3.5. *It is noted that the graph G mentioned in Result 3.4 is the complete bipartite graph* $K_{1,n}$ *which is known as a star.*

THEOREM 3.6.

$$g_{it}(K_{p,q}) = \begin{cases} 3 \ \ if \quad p = 2 \text{ and } q \geq 2 \\ 4 \qquad if \quad p, q \geq 3 \end{cases}$$

PROOF. The complete bipartite graph $K_{p,q}$ is the one whose vertex set is divided into two non-overlapping sets, $V_1 = \{x_1, x_2, ..., x_p\}$ and $V_2 = \{y_1, y_2, ..., y_q\}$.
Case 1: $p = 2$ and $q \geq 2$
Then $V_1 = \{x_1, x_2\}$ and $V_2 = \{y_1, y_2, ..., y_q\}$
For $q > 2$, V_2 (of size q) is the unique β_0-set.
If $q = 2$, then both V_1 and V_2 are of equal size and are independent sets. So, there are two β_0-sets V_1 and V_2 in $K_{p,q}$.
Now, define $S = \{x_1, x_2, y_i\}$ where $y_i \in V_2$ for any $i = 1, 2, ..., q$.
For $q > 2$, the only β_0-set is V_2 and $y_i \in S \cap V_2$, so it's a transversal.
If $q = 2$, the β_0-sets are V_1 and V_2. Since $x_1, x_2 \in V_1$ and $y_i \in V_2$, S intersects both.
In $K_{p,q}$, all shortest paths are of length 1 or 2, since any two vertices in different partitions are adjacent (distance 1), and any two vertices in the same partition are at distance 2 (going through a vertex in the opposite partition).
Now, since $S = \{x_1, x_2, y_i\}$ includes two vertices from V_1 and one from V_2, the shortest paths between all three elements cover all the vertices of the graph. That is, x_1-y_i-x_2 includes all of V_1. Also, all the vertices of V_2 are included via the shortest paths from x_1 to x_2.
$\therefore S$ is a g_{it}-set in $K_{p,q}$ and so $g_{it}(K_{p,q}) = 3$.
Case 2: $p, q \geq 3$
If $p = q$, then both V_1 and V_2 are β_0-sets of size p in $K_{p,q}$.
If $p \neq q$, then the larger of the two (either V_1 or V_2) is the unique β_0-set in $K_{p,q}$.
Now define $S = \{x_i, x_j, y_k, y_l\}$ for any $i, j = 1, 2, ..., p$ and $k, l = 1, 2, ..., q$ with $i \neq j$ and $k \neq l$.
Since it contains exactly 2 vertices from both V_1 and V_2, it intersects both.
With 2 vertices in each partition, every vertex in V_1 lies on a path between two vertices in V_2 and vice versa.
For eg., $x_m \in V_1$ lies on a path between y_k and y_l: y_k-x_m-y_l. Similarly, $y_n \in V_2$ lies on a path between x_i and x_j: x_i-y_n-x_j.
Thus S is a g_{it}-set in $K_{p,q}$.
So $g_{it}(K_{p,q}) = 4$. □

REMARK 3.7. *g(*$K_{r,s}$*) = min {4, r} where* $2 \leq r \leq s$ *[7].*

THEOREM 3.8. $g_{it}(P_n) = 2$ *where* P_n *is a path with* $n \geq 3$*.*

PROOF. Assume $V(P_n) = \{x_1, x_2, ..., x_n\}$.
$I = \{x_1, x_3, ..., x_n\}$ is the unique β_0-set in the path graph P_n when n is odd.
$J_1 = \{x_1, x_3, ..., x_{n-1}\}$, $J_2 = \{x_2, x_4, ..., x_n\}$, $J_3 = \{x_1, x_4, x_6, ..., x_n\}$ and $J_4 = \{x_1, x_3, ..., x_{n-3}, x_n\}$ are the β_0-sets in P_n when n is even.
Define $S = \{x_1, x_n\}$. Then S is a geodetic set intersecting I when n is odd.
When n is even, S intersects all the β_0-sets J_1, J_2, J_3 and J_4 mentioned above.
Thus S is a g_{it}-set in P_n and so $g_{it}(P_n) = 2$.
□

THEOREM 3.9. *For the even cycles* C_{2m}*,*

$$g_{it}(C_{2m}) = \begin{cases} 2 \ \ if \quad m \quad is \quad odd \\ 3 \ \ if \quad m \quad is \quad even \end{cases}$$

PROOF. Suppose $V(C_{2m}) = \{x_1, x_2, ..., x_{2m}\}$. Then there are exactly two β_0-sets $I_1 = \{x_1, x_3, ..., x_{2m-1}\}$ and $I_2 =$

$\{x_2, x_4, ..., x_{2m}\}$ of C_{2m}.
When m is odd
Let $S_i = \{x_i, x_{i+m}\}, i = 1, 2, ..., m$
It is easy to verify that each S_i is a geodetic set intersecting both I_1 & I_2. $\therefore$, each S_i is a g_{it}-set.
Hence, $g_{it}(C_{2m}) = 2$.
When m is even
Let $S_i = \{x_i, x_{i+1}, x_{i+m}\}, i = 1, 2, ..., m$
Then each S_i intersects both β_0-sets I_1 & I_2 and is also a geodetic set. $\therefore$, each S_i is a g_{it}-set.
Hence, $g_{it}(C_{2m}) = 3$. □

THEOREM 3.10. *For an odd cycle* C_{2m+1},

$$g_{it}(C_{2m+1}) = \begin{cases} 3 & when \quad m \quad is \quad odd \\ 4 & when \quad m \quad is \quad even \end{cases}$$

PROOF. Assume $V(C_{2m}) = \{x_1, x_2, ..., x_{2m+1}\}$.
C_{2m+1} has exactly $2m+1$ β_0-sets, each consisting of m nodes.
Let these β_0-sets be labeled as I_j where $j = 1, 2, ..., 2m+1$.
When m is odd
Define $S_i = \{x_i, x_{i+1}, x_{i+m+1}\}, i = 1, 2, .., m$. Then each S_i is a geodetic set.
It has to be proved that $S_i \cap I_j \neq \phi$ for all $j = 1, 2, ..., 2m+1$.
Suppose, for contradiction that $S_i \cap I_j = \phi$ for some j.
Then $x_i, x_{i+1}, x_{i+m+1} \notin I_j$.
By removing the three appropriate chosen vertices from the cycle graph C_{2m+1}, the induced subgraph on the rest of the $2m-2$ vertices becomes disconnected, consisting of two disjoint paths, each containing $m-1$ vertices. For each path, the largest possible set of independent vertices has a size of $\frac{m-1}{2}$ since it has an even number of vertices $m-1$. Thus I_j contains at most $m-1$ independent vertices.
This contradicts the fact that any β_0-set in C_{2m+1} must contain exactly m nodes.
$\therefore S_i \cap I_j \neq \phi$ for all j.
Hence each S_i is a g_{it}-set.
$\therefore g_{it}(C_{2m+1}) = 3$.
When m is even
Let $S_i = \{x_i, x_{i+1}, x_{i+m+1}, x_{(i+m+2) mod(2m+1)}\}, i = 1, 2, ..., m$. Then each S_i is a geodetic set.
We claim that S_i intersects every β_0-set I_j of C_{2m+1} where $j = 1, 2, ..., 2m+1$.
Suppose, for contradiction that $S_i \cap I_j = \phi$ for some j.
Then $x_i, x_{i+1}, x_{i+m+1}, x_{(i+m+2) mod(2m+1)} \notin I_j$.
Remaining $2m-3$ vertices of the cycle induce a disconnected sub graph containing two disjoint paths: one path with $m-1$ vertices and another with $m-2$ vertices.
The maximum number of independent vertices in these paths is $\frac{m-1}{2} + \frac{m-2}{2} = \frac{2m-3}{2}$ which is strictly less than m (since m is even and $\frac{2m-3}{2} = m - \frac{3}{2}$).
Hence, I_j contains fewer than m independent nodes contradicting the fact that any β_0-set in an odd cycle C_{2m+1} must contain exactly m nodes.
$\therefore S_i \cap I_j \neq \phi \, \forall \, j$.
So each S_i is a g_{it}-set.
$\therefore g_{it}(C_{2m+1}) = 4$. □

THEOREM 3.11. *For the hypercube* Q_m *with* $m \geq 3$,

$$g_{it}(Q_m) = \begin{cases} 2 & if \quad m \quad is \quad odd \\ 3 & if \quad m \quad is \quad even \end{cases}$$

PROOF. Q_m is an m-regular graph with 2^m vertices which are represented by binary strings (0 and 1) of length m. Edges connect two vertices in Q_m whose strings differ in only one bit.
For any $x \in Q_m$, let x^c denote its complement, obtained by replacing every 0 in x with a 1, and every 1 with a 0.
The weight of a vertex in the m dimensional hypercube Q_m is the number of ones in its binary representation. Thus the vertices are equally divided, with (2^{m-1}) having an odd weight and (2^{m-1}) having an even weight. So every edge in Q_m connects a vertex of even weight to a vertex of odd weight. This means the hypercube is a bipartite graph. Let I_1 ={vertices of even weight} and I_2 ={vertices of odd weight}. Then I_1 and I_2 forms the bipartition of the vertex set of Q_m.
Moreover, the only two β_0-sets in the hypercube graph Q_m are I_1 and I_2.
When m is odd
Let $S = \{x, x^c\}$, for any $x \in Q_m$
A geodesic of length m contains an even number of internal vertices. So if we place two vertices x and x^c in the set S, then all vertices along their geodesic are covered as intermediate vertices on shortest paths between elements of S. Hence, all vertices of Q_m lie on some geodesic between x and x^c, and so $S = \{x, x^c\}$ is a geodetic set.
For any vertex x in I_1, its complement x^c is in I_2 since m is odd.
It follows that S intersects both I_1 and I_2.
Thus S is a g_{it}-set in Q_m.
Hence $g_{it}(Q_m) = 2$.
When m is even
Suppose, $x \in I_1$, since m is even, $x^c \in I_1$. Similarly, if $x \in I_2$, obviously $x^c \in I_2$.
Consider the set $S = \{x, y, x^c\}$ where x and y are adjacent in Q_m.
It follows that if $x \in I_1$, then $y \in I_2$ and conversely.
It is clear that every vertex in Q_m lies on a shortest path between some pair of vertices in S. Then S is a geodetic set intersecting both of the β_0-sets I_1 and I_2. Thus S is a g_{it}-set of Q_m in this case.
Hence $g_{it}(Q_m) = 3$. □

THEOREM 3.12. *The set of all end vertices in a tree* T *forms a minimum independent transversal geodetic set and* $\therefore g_{it}(T) = k$.

PROOF. It is obvious that in a tree T, at least one of the end vertices is included in every β_0-set I.
Let the set of all end vertices in T be denoted by S. Then $|S| = k$ since T has k end vertices .
Also by theorem 1.3, S is a geodetic set of minimum cardinality.
It is evident that S necessarily intersects every β_0-set I in T.
Thus S is a g_{it}-set in T.
$\therefore g_{it}(T) = k$. □

4. BOUNDS OF g_{it} AND EXISTENCE THEOREM ON g_{it}

THEOREM 4.1. *For any simple connected graph* G *on* n *vertices, the independent transversal geodetic number satisfies the inequality* $2 \leq g_{it}(G) \leq n$.

The next Theorem is a direct outcome of Theorem 1.1[4].

THEOREM 4.2. *Every independent transversal geodetic set in* G *consists of all of its extreme vertices. This includes all end vertices, which are a specific type of extreme vertex.*

The next corollary follows from Theorems 4.1 and 4.2.

COROLLARY 4.3. *If* G *is a simple connected graph on* n *vertices with* p *extreme vertices (or end vertices), then the indepen-*

dent transversal geodetic number $g_{it}(G)$ *satisfies the inequality:* $\max(2, p) \leq g_{it}(G) \leq n$

THEOREM 4.4. *Let r, d and m be positive integers with m* ≥ 2 *and r* $\leq$ *d* $\leq$ *2r. Then there exists a connected graph G such that the radius of G is r, diameter of G is d and* $g_{it}(G)$ *= m.*

PROOF. For r = 1, two cases are considered for the value of d: d = 1 and d = 2.
For d = 1, let G be complete graph K_m. According to Theorem 3.1, $g_{it}(G) = m$.
For d = 2, consider the star graph G = $K_{1,m}$ where m $\geq$ 2. From Result 3.4, it is established that $g_{it}(G)$ is equal to m.
Now suppose that r $\geq$ 2. Then there are two cases r = d and r $<$ d.
Case 1: r = d.
If m = 2, 3, 4, then there exists an even or odd cycle which satisfy the necessary requirements for any value of r.
For m $\geq$ 5, we proceed as follows.
First suppose that r = 2.
Construct the graph G by starting with the 4-cycle C_4 :v_1 v_2, v_3, v_4, v_1 and adding $m - 2$ new vertices x_1, x_2, ..., x_{m-2}, each adjacent to both v_1 & v_2.
The graph G corresponding to $m = 5$ is presented in Figure 3.

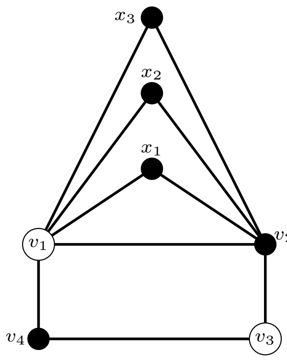

Fig. 3. Graph G with r = d = 2 and g_{it}(G) = 5

Then the β_0-sets in G are I_1 =$\{x_1, x_2, ..., x_{m-2}, v_4\}$ and I_2 = $\{x_1, x_2, ..., x_{m-2}, v_3\}$.
Now S =$\{x_1, x_2, ..., x_{m-2}, v_2, v_4\}$ is a minimum geodetic set intersecting I_1 and I_2. Hence S forms a g_{it}-set in G and $g_{it}(G)$ = m.
It is observed that $\{x_1, x_2, ..., x_{m-2}, v_1, v_3\}$ and $\{u_1, u_2, ..., u_{m-2}, v_3, v_4\}$ can also be identified as g_{it}-sets in G.

Now let r = 3. Then construct the graph G as follows:
Begin with the 6-cycle C_6 : x, u, y, v, z, w, x. Then perform the following steps:
(1)Add $m - 3$ new vertices u_1, u_2, ..., u_{m-3}, each joined to both x & y
(2)Add $m - 3$ new vertices w_1, w_2, ..., w_{m-3} each joined to both to x & z
(3)Add a final set of $m - 3$ new vertices x_1, x_2, ..., x_{m-3} where each x_i is joined to u_i and w_i for $i = 1, 2, ..., m - 3$.
Figure 4 illustrates the graph G for the case when $m = 5$.

Let S_1 = $\{u, u_1, u_2, ..., u_{m-3}\}$, S_2 = $\{v\}$ and S_3 = $\{w, w_1, w_2, ..., w_{m-3}\}$.

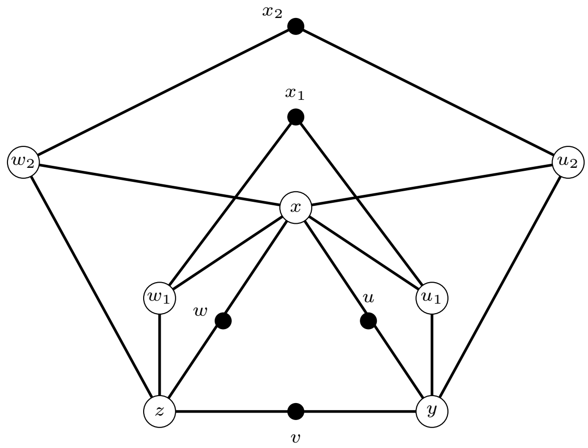

Fig. 4. Graph G with r = d = 3 and g_{it}(G) = 5

Then I = $S_1 \cup S_2 \cup S_3$ is the unique β_0-set in G.
It is clear that S=$\{x_1, x_2, ..., x_{m-3}, u, v, w\}$ is a g-set of G, intersecting I.
This implies that S is a g_{it}- set yielding $g_{it}(G) = m$.

Next, consider the case r = 4.
For each integer i satisfying $1 \leq i \leq m - 4$, define F_i = $\{u_{i1}, u_{i2}\}$ and H_i = $\{w_{i1}, w_{i2}\}$ to be two distinct copies of the path graph P_2.
Construct G as follows:
Start with the 8-cycle C_8:v_1, v_2, ..., v_8, v_1.
Then perform the following additions:
(1)For each $i = 1, 2, ..., m - 4$, join u_{i1} to v_2 and u_{i2} to v_4
(2)For each $i = 1, 2, ..., m - 4$, join w_{i1} to v_8 and w_{i2} to v_6
(3)Then add $m - 4$ new vertices x_1, x_2, ..., x_{m-4} and join x_i with u_{i1} & w_{i1} for each $i = 1, 2, ..., m - 4$.

An illustration of the graph G for the specific case r = 4 & m = 6 is provided in Figure 5 .

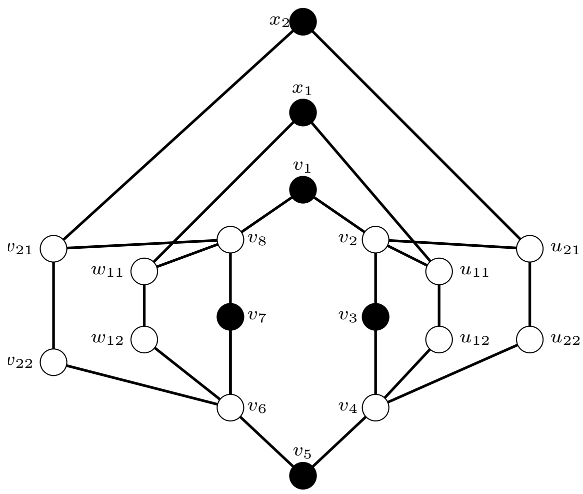

Fig. 5. Graph G with r = d = 4 and g_{it}(G) = 6

Let $S_1 = \{u_{12}, u_{22}, ..., u_{(m-4)2}$ $\}$, $S_2 = \{w_{12}, w_{22}, ..., w_{(m-4)2}$ $\}$, $S_3 = \{x_1, x_2, ..., x_{m-4}\}$ and $S_4 = \{v_1, v_3, v_5, v_7\}$.
Then it is obvious that I $=S_1 \cup S_2 \cup S_3 \cup S_4$ is the unique β_0-set of the graph G.
Now S = $\{v_1, v_3, v_5, v_7, x_1, x_2, ..., x_{m-4}\}$ is a g-set intersecting I. therefore S is a g_{it}-set and so $g_{it}(G) = m$.

Suppose r $\geq$ 5, we consider the following two sub cases.

Subcase 1.1: Let $m = 2p + 1 \geq 3$ be an odd integer.
For each positive integer i with $1 \leq i \leq 2p - 1$, define a path P_i consisting of the nodes $z_{i1}, z_{i2}, ..., z_{i(2r-5)}$, such that P_i is identical to P_{2r-5}.

To construct the graph G, we begin with the even cycle C_{2r}: $v_1, v_2, ..., v_{2r}, v_1$ and join z_{i1} to v_{2r} and $z_{i(2r-5)}$ to v_2 for each $i = 1, 2, ..., 2p - 1$.

For instance, when r = d = 5 & m = 5, the graph G appears as in Figure 6.

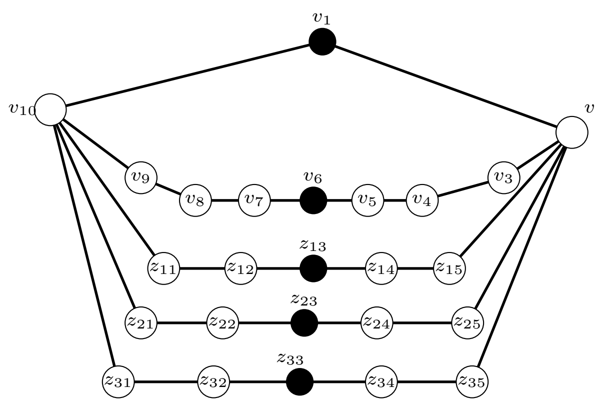

Fig. 6. Graph G with r = d = 5 and g_{it}(G) = 5

Now let $S_i = \{z_{i1}, z_{i3}, ..., z_{i(2r-5)}$ $\}$ for $i = 1, 2, ..., 2p - 1$ and $T = \{v_1, v_3, ..., v_{2r-1}\}$. Then obviously I $=S_1 \cup S_2 \cup ... S_{2p-1} \cup T$ is the unique β_0-set of G.
Define $S = \{z_{i(\lfloor \frac{r}{2} \rfloor+1)}; (1 \leq \text{i} \leq 2p - 1)\} \cup \{v_1, v_{r+1}\}$. Then S is a g_{it}- set.
$\therefore$, $g_{it}(G) = 2 + 2p - 1 = 2p + 1 = m$.

Subcase 1.2: $m = 2p + 2$ $\geq$4 is even.
Form the graph G by extending the graph considered in Subcase 1.1 (for $m = 2p + 1$) by including a vertex u and joining it to the vertices v_2 and v_{2r}.
Example graph for r = d = 5 & m = 6 is as depicted in Figure 7.

Now the set I considered in Subcase 1.1 along with the vertex u becomes the unique β_0-set in G.
That is, let $J = I \cup \{u\}$. Then J is the unique β_0-set in G.
Also $S' = S \cup \{u\}$ where S is the set considered in Subcase 1.1, Then S' necessarily intersects J and is a g_{it}-set.
Hence $g_{it}(G) = 2p + 2 = m$.

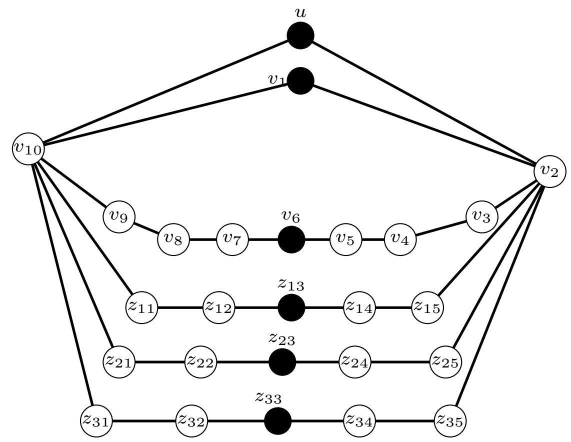

Fig. 7. Graph G with r = d = 5 and g_{it}(G) = 6

Case (2): Consider r $<$ d.
C_{2r} is an even cycle with vertices v_1, v_2, ..., v_{2r}, where the edges form a closed loop: C_{2r} $=(v_1, v_2, ..., v_{2r}, v_1)$. Let P_{d-r+1} be a path with vertices: u_0, u_1, u_2, ..., u_{d-r}.
Construct a graph H by identifying vertex v_1 from the cycle with vertex u_0 from the path, effectively merging the two graphs at this common vertex.
Next, create a graph G by adding $m - 2$ new vertices w_1, w_2, ..., w_{m-2} and connecting each of them to vertex u_{d-r-1} in the path.
The pictorial representation of G thus obtained is as in the following Figure 8.

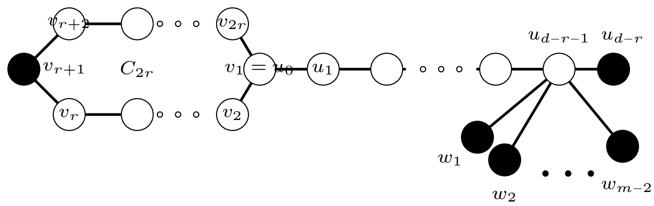

Fig. 8. Graph G of radius r and diameter d with $r < d$ & g_{it}(G) = m

It is evident from the construction of the graph G that its radius is r and diameter is d with $r < d$.
Let S = $\{u_{d-r}, w_1, w_2, ..., w_{m-2}\}$ be a set consisting of $m - 1$ end vertices of the graph G. By Theorem 1.1, it follows that S is necessarily a subset of every geodetic set in G. Moreover, S is contained in every β_0-set of G.
Still, the set S is not geodetic, since the shortest paths between its vertices do not cover all vertices of G.
Consider the set S' = $S \cup \{v_{r+1}\}$. This set forms a geodetic set of minimum cardinality, since the addition of v_{r+1} in S ensures that all vertices in G are included in some shortest path between any pair of vertices in S'.
Furthermore, S' cuts every β_0-set, making it a g_{it}-set.
Thus, we conclude that $g_{it}(G) = m$. $\square$

REMARK 4.5. *It can be noted from the above Theorem 4.4 that there always exists a connected graph G with specified independent transversal geodetic number* $m \geq 2$.

5. RELATION BETWEEN $g(G)$ AND $g_{it}(G)$

In this section, the relation between geodetic number and independent transversal geodetic number is established. Based on the observations and ideas from the previous sections, the graphs G wherein the two parameters $g(G)$ & $g_{it}(G)$ are identical and distinct are characterized.

THEOREM 5.1. *For any simple connected graph G, $g_{it}(G) \geq g(G)$.*

PROOF **Proof:.** Let S be a g-set in G with cardinality m.
If S intersects every β_0-set in G, then S itself is a g_{it}-set of cardinality m.
So $g(G) = g_{it}(G)$.
If S does not intersect at least one β_0-set in G, then S is not an independent transversal geodetic set.
Suppose that S does not intersect k maximum independent sets I_j ; $j = 1, 2, ..., k$.
Let $S' = S \cup \{u_1, u_2, ..., u_k\}$ where $u_1 \in I_1, u_2 \in I_2, ..., u_k \in I_k$.
Then S' intersects every β_0-set in G.
If $k = 1$, then $|S'| = m + 1$ and if $k > 1$, then $|S'| = m + k$. But if $u_i = u_j$ for some $i \neq j$, then $|S'| < m + k$. Anyway, the cardinality of S' is at least $m + 1$.
Thus S' is a g_{it}-set and $g_{it}(G) \geq m + 1 > m = g(G)$.
That is, $g_{it}(G) > g(G)$.
Hence $g_{it}(G) \geq g(G)$. □

THEOREM 5.2. *If G is a simple connected graph with at least one end vertex, then $g_{it}(G) = g(G)$.*

PROOF. Let S be a g-set in G. Then $g(G) = |S|$.
By Theorem 1.1, all the end vertices of G belong to S.
Also by Theorem 4.2, each end vertex of G must be included in every independent transversal geodetic set of G and so S itself is such a set.
Hence S is a g_{it}-set in G.
$\therefore g_{it}(G) = |S| = g(G)$. □

REMARK 5.3. *It can be noted that the converse part of the above Theorem 5.2 is not true generally. That is, if $g_{it}(G) = g(G)$, then the graph G need not have any end vertex.*
For instance, in Example 2, the graph G has no end vertex. But it is evident that $g_{it}(G) = g(G)$.

THEOREM 5.4. *Suppose G is a graph having more one β_0-set. Then $g_{it}(G) > g(G)$ iff there does not exist a g-set intersecting every β_0-set in G.*

PROOF. Assume that $g_{it}(G) > g(G)$.
To prove: There does not exist a g-set intersecting every β_0-set in G.
Now suppose that there exists a g-set S intersecting every β_0-set in G.
Then S itself is a g_{it}-set in G.
This implies that $g_{it}(G) = g(G)$ which is a contradiction.
$\therefore$ there does not exist a g-set intersecting every β_0-set in G.
Conversely, assume that there does not exist a g-set intersecting every β_0-set in G.
To prove: $g_{it}(G) > g(G)$.
Suppose to the contrary that $g_{it}(G)$ is not greater than $g(G)$. Then by Theorem 5.1, $g_{it}(G) = g(G)$.
$\therefore$ it is possible to find a g-set which is also a g_{it}-set. That is, there exists a g-set intersecting every β_0-set in G. This is a contradiction to our assumption.
Hence $g_{it}(G) > g(G)$. □

REMARK 5.5. *It is discerned that a graph G having unique β_0-set I may admit a g-set that either intersects I or does not. In the former case, $g_{it}(G) = g(G)$, while in the latter, $g_{it}(G) > g(G)$. Typically, in most graphs, the g-set intersects I, leading to $g_{it}(G) = g(G)$. However, there exist graphs where the g-set does not intersect I resulting in $g_{it}(G) > g(G)$. The following examples illustrate this distinction effectively.*

EXAMPLE 3. *Let us look at the following Figure 9.*

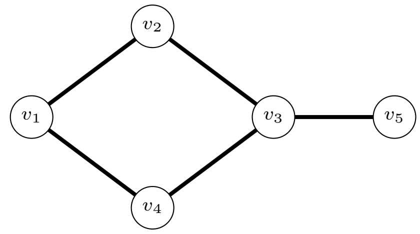

Fig. 9. Graph G

The sets $I = \{v_2, v_4, v_5\}$ is the unique β_0-set in G.
$S = \{v_1, v_5\}$ is a geodetic set of minimum cardinality which also intersects I.
S is a g_{it}-set.
Hence $g_{it}(G) = 2 = g(G)$ in this graph.

EXAMPLE 4. *Let G be the graph depicted in Figure 10.*
The set $I = \{v_2, v_4, v_5, v_7\}$ is the unique β_0-set in G.
$S = \{v_1, v_6\}$ is a geodetic set of minimum cardinality which does not intersect I.

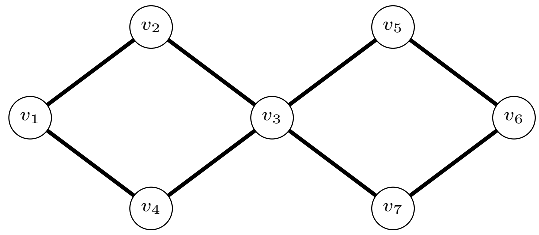

Fig. 10. Graph G

So let $S' = \{v_1, v_2, v_6\}$.
Then S' intersects I and so is a g_{it}-set.
Hence $g_{it}(G) = 3$, but $g(G) = 2$.
Thus $g_{it}(G) > g(G)$ in this graph.

6. SCOPE

This article introduces and analyzes the concept of independent transversal geodetic number of a graph. There is a scope for further study on the necessary conditions for a graph to have equal geodetic number and independent transversal geodetic number. Also, further investigations can be carried out to find the relationship between g-sets and g_{it}-sets in a graph.

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