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Exploring Fuzzy Filters and Ideals in BG-Algebras: A Comprehensive Framework for Maximal Extensions and Algebraic Generalizations

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ABSTRACT

The work incorporates fuzzy set theory in the context of BG-algebras, defining fuzzy filters and ideals which allow the preservation of operations of BG-algebras despite partial membership. This consolidation is vital to ensure that classical algebraic fundamentals remain relevant even in the context of fuzziness. Furthermore, the work provides such a theory for the more advanced maximal fuzzy filters, whose closure properties are essential in the broader design of the theory with Zorn's Lemma. It also investigates mapping properties and provides numerous examples of fuzzy filters, ideals and complemented fuzzy filters, symmetric and bounded fuzzy filters and ideals. The paper also introduces fuzzy prime filters that are a tool for re-evaluating the algebraic behavior of BG-algebras. The results provided by this paper gives a complete framework to understand all BG-algebras as well as a generalization of them in fuzzy algebraic systems. This theoretical framework not only contributes to the field of algebra, but may also serve as a precursor for applications in domains like fuzzy logic, decision-making, and computing systems. The further investigation of closure properties, maximal extensions and structural relationships may pave an interesting path for both theory and practice.

Keywords

BG-algebra, fuzzy filters, fuzzy ideals, maximal fuzzy filters, fuzzy prime filters

1. INTRODUCTION

Abstract algebras find use in more than simple mathematical structures, providing context for how relationships, transformations, and operations are expressed and examined within abstract systems. Among these algebraic systems, *BCK-algebras* and *BCI-algebras* are two kinds of more primitive algebraic systems. They were first presented as algebras in order to generalize some structures in logic and algebra, where a BCI-algebra was recognized as a non-trivial subclass of a BCK-algebra. With the works of Q. P. Hu and X. Li [12, 13] the other algebraic structures appeared: *BCH-algebras*. BCH-algebras were introduced as a natural proper sub-class of BCI-algebras [15], leading to a structure where algebraic systems that preserve more generalized properties tower over each other, while containing safely their predecessors in one class. This step opened the door to new, much wider frameworks for mathematicians working in many different areas of logic and algebra. Part of this increasing corpus of work was done by J. Neggers and H. S. Kim in [31], in which they introduced *d-algebras*. An even broader extending object than BCK-algebras [16, 28] are called *d-algebras*. The idea of *B-algebras* was proposed [32] based on d-algebras. B-algebras are an algebraic class defined with a set of specific elements and structural constraints that offer a unique framework for further analysis. With new algebraic items and operations B-algebras provide further ways of investigating properties which are not accessible using BCK and BCI-algebras, the varieties they simplify. Y. B. Jun, E. H. Roh and H. S. Kim generalized the family of B-algebra structures to a larger family of B-algebra structures and called them by BH-algebras [18]. The BH-algebras they introduced generalized BCK-, BCI-, and BCH-algebras, including all of them as subclasses, but also broadened the principles behind them as well. This series illustrated the breadth of algebraic generalization, where new systems are built on basic properties but with more encompassing axioms, yielding more new algebraic applications and more algebraic theory. Most recently, C. B. Kim and H. S. Kim considered also *BG-algebras* [21], a further step into abstraction. BG-algebras generalize B-algebras, but with more operations and extra equations that set them apart from previous classes and develop them to an actual algebraic structure. BG-algebras are a new rung on the ladder of the hierarchy of abstract algebraic structures, but they reflect an ongoing tendency in the mathematical modeling enterprise to devise more abstract and generalizable frameworks. The developing algebraic structures of BCK and BCI-algebras to BG-algebras have the above characterization of being both inclusive and generative of prior structures evidently emphasizing the logics in which a general algebraic theory of algebraic types arises over an algebraic basis modeling serviceable utility across theoretic areas of mathematics.

Fuzzy set theory, first proposed by Zadeh as a fundamental constituents of fuzzy logic through an axiomatic framework in the 1960s [41], has endeavored to close the chasm between classical algebraic structures and fuzzy logic. Fuzzy algebraic structures have been a blooming area of research due to the work of the first pioneers A. Rosenfeld [34] and many others like W.-J. Liu [22, 23] works laid the foundation for establishing connections between classical algebraic structures and fuzzy set theory, giving rise to a fruitful development of the theory. Rosenfeld set the foundation for fuzzy sets in algebraic systems, and Liu has also contributed to the theoretical foundations of fuzzy algebraic operations. They rightfully received credit for initiating the field of fuzzy algebra which incorporates algebraic concepts from diverse branches of mathematics such as group theory, ring theory and lattice theory. After this set of pioneering developments, the work of early extensions of fuzzy set into certain special algebraic structures was carried out, for example, by D. S. Malik, J. N. Mordeson [24], and S.-R. Shi [37] on those algebraic structures. They came up with fuzzy subrings in addition to fuzzy subgroups leading to their concepts being indispensable in the extract of how fuzzy algebraic systems can be fuzzyfied. In these studies, the fuzzy subset notion is generalized into algebraic entities like subring and subgroup?they offered a more advanced level of abstraction for algebraic scrutiny. Fuzzy set theory has greatly contributed to the understanding and development of algebraic structures such as BCK and BCI-algebras. In 1991, Xi [38] first introduced the notions of fuzzy sets in BCK-algebras, and the new concept of fuzzy sets interacts with the natural operations of BCK-algebras, which led to the direction of studying BCK-algebras with partial ordering structure. BCI-algebras were soon generalized to enter the framework of fuzzy set theory by Ahmad [1] and Jun [17] in 1993. One of the most explored topics that appeared from these insights is the theory of fuzzy ideals. For this reason, when dealing with BCK and BCI-algebras, fuzzy ideal theory focuses on the natural question which is that how a fuzzy subset of an algebra can be used to generate an ideal in the given algebra. In algebraic structures, ideal theory is important as it defines almost closed sub-structures with respect to given operations. Only in classical algebra, ideals are exactly what you need to build quotients and study homomorphisms. It is only a simple glance on the mathematical background that some of these ideas can be extended in what concern the notion of fuzzy sets but it is not so trivial as for a very small glance this is because generate ideal and homomorphisms cannot be defined in the classical sense since fuzzy subsets are defined in non-binary value membership but $\phi \in [0,1]$ which calls for additional definitions based on clever representations rather than looking over classical concepts. The concepts of fuzzy ideals in BCK and BCI-algebras have been studied by Meng and Guo [26], which prove these structures can be extended to accommodate fuzzy input and output. More complete works on fuzzy ideal theory on these algebras build on their works. Research on fuzzy ideal theory in BCK/BCI-algebras is still in development, and its implications are significant for the generalization of algebraic structures and the solving of real-world problems, particularly those involving uncertainty and incompleteness of information. In the year 1999, given the idea of fuzzy subalgebras in the context of BG-algebras have been introduced by Ahn and Lee [2]: this has been a starting point to increase the influence of fuzzy sets in the area of algebraic systems. The notion of fuzzy subalgebras provides an additional degree of flexibility to modeling uncertain or imprecise data since it introduces an uncertainty level on the standard properties of a subalgebra (for example, closure and homomorphism) to be preserved under fuzzification. In the recent paper, Muthuraj et al [29] studied the fuzzification of ideals in BG-algebras with detailed discussion on the nature of such new fuzzy ideals and their relationship with the classical ideals. Huang [14] introduced the concept of the ideals and extended the theory by studying the fuzzification of some operations in the BG-algebras and its new results of algebraic and topological properties. Hong and Jun [11] further work on fuzzy ideals where they generalized Huang?s definition and presented doubt fuzzy ideals in BCK and BCI-algebras. Its significance came from the introduction of fuzzy ideal theory: it provided a new method of integrating doubt into fuzzy ideal theory, which therefore allowed more interrelations with fuzzy logic in algebraic structures. In addition, Barbhuiya [5], provided an enhancement in the fuzzy ideals, by defining the (α, β)-doubt fuzzy ideals in BG-algebras, thus, introducing a new era in the application of fuzzy logic in algebra. These events are an indication of the maturity of fuzzy algebraic structures and fuzzy algebraic systems. Fuzzy Algebra since then has been able to extend classical algebraic concepts like ideals and subalgebras into the fuzzy realm, allowing for having additional advanced algebraic tools to be applicable to difficult problems in logic, computer science and decision-making which naturally involve uncertainty and imprecision.

The generalization of BCK and BCI -algebras with fuzzy logic structures have developed, making a linking between abstract algebra and applied computational frameworks. M. Akram and A. Al-Masarwah [3, 4] have made contributions to the field, examining m- polar fuzzy structures and proving their algebraic properties in the context of BCK and BCI-algebras. Akram has raised the importance of fuzzy structures in both theoretical and applied settings, mentioning fuzzy hypergraphs and hyperalgebras that are practical tools for modeling information systems and computational frameworks. Al-Masarwah has introduced changes in bipolar fuzzy ideals and subalgebras, which shows how these structures can be adjusted to computational settings, when operations can be extended through bipolar fuzzy elements. It appears that such an extended set of operations allows improving the algebraic process in applications of duality and symmetry. Further understanding of fuzzy dot structures of B-algebras and BG-algebras by T. Senapati and M. Pal [36] has proven enormous potential for generalization within fuzzy logic systems. The researchers have raised essential topics like fuzzy dot ideals and interval-valued intuitionistic fuzzy sets, showing the development in uncertainty modeling that is an essential asset for artificial intelligence and fuzzy reasoning systems where reasoning under incomplete information is a core competency. The combination of abstract theory and practice applications not only has enriched the theoretical field of BCK/BCI-algebras but also has vastly developed the use of fuzzy algebras in such advanced areas as database theory, artificial intelligence, and decision support systems. It seems that the obtained implications show the robustness of fuzzy algebraic structures in computational sciences where precise treatment of uncertainty and variability is of the highest importance. The development of these topics raises new perspectives for the future of fuzzy logic, which seems to be rapidly changing and gaining relevance in modern informatics. The studies on fuzzy BG-algebras have raised considerable interest in the context of fuzzy logic, particularly raised by fuzzy ideals, filters, and their generalization within the algebraic structure of BG-algebras. Several types of fuzzy ideals have been studied, including $(\in, \in)$-fuzzy ideals, $(\in, \in \vee q)$-fuzzy ideals, and (r, l)-derivations in BG-algebras. The properties and operations relying on BG-algebras are crucial for fuzzy logic applications since they help distinguish uncertain or imprecise elements of information in decision support systems and data analysis. D.K. Basnet and L.B. Singh [6] have raised the nuances of $(\in, \in \vee q)$-fuzzy ideals, proposing how they can be defined and showing their core differences with regular $(\in, \in)$-fuzzy ideals. Moreover, the conditions when a fuzzy subset of a BG-algebra can be viewed as a $(\in, \in \vee q)$-fuzzy ideal have been shown, proving the convenient and functional depth of BG-algebra structures for representing varying degrees of membership and interaction in fuzzy logic systems. The research on (l,r)-derivations and left derivations in BG-algebras by Kamaludin, Sri Gemawati, and Kartini [20] has explored the properties that make the algebraic structure fitting for analyzing logical derivations in the fuzzy logic context. It is apparent that the specific algebraic properties of BG-algebras have proven to be suitable for advanced fuzzy systems, significantly contributing to unrecognized potential of fuzzy algebraic structures in the context of handling vague and uncertain data in informatics and applied mathematics.

Fuzzy filter theory has made significant implications in pure mathematics. The application of fuzzy filter theory has been essential in lattice theory, algebraic structures, topology, and logic. Lattice theory is a defining topic in pure mathematics as far as the realization of fuzzy filter theory implications is concerned. The concept of a fuzzy filter extends the concept of a filter in lattice theory to a fuzzy setting. Fuzzy filters have been useful for exploring lattice-valued functions, as well as their properties. Other topics that have been useful for study include the theory of fuzzy topological spaces, fuzzy ideals in rings and algebras. The exploration of fuzzy filter concepts has been useful in lattice structures for understanding closure operators and the structure of fuzzy relations in complete lattices and Heyting algebras. According to previous research by Liu [10] and Yager [40], fuzzy filters are used to extend the classic results related to filters in these settings to fuzzy filters and operations. In algebraic structures such as BG-algebras, BCK-algebras, and BCI-algebras, the implications of fuzzy filter theory have been useful. Fuzzy filters have been useful as defined in the works of Meng [27] in BG-algebras to study fuzzy subalgebras and homomorphisms. Fuzzy filter implications exist in the topology topics that enhance the study of fuzzy topological spaces [30]. This work has been useful for studying continuity, convergence, and compactness. Fuzzy filters have been used to define fuzzy open sets, and concepts of fuzzy compactness have been defined. One of the temperatures has been used to study the extent to which topological concepts can be translated into a fuzzy environment. Implications of fuzzy filters in mathematical logical approaches are profound, especially in fuzzy logic systems. Fuzzy filters have been useful in letting classical logical operations conceptions give with ease. Past applications of the filters in the development of fuzzy logic as the study of uncertainty and vagueness show were useful in the work of Mendel [25] and Zadeh [42]. Implications of fuzzy measures and integrals have been studied, and a necessary extension was made, referred to as fuzzy. These concepts have been useful in decision-making and optimization, especially with uncertainty involved. In conclusion, fuzzy filter theory has had implications in pure mathematics. The topic has been useful in extending most classical topics of study, and it has provided useful tools for handling logical uncertainty and imprecision.

Over the past 30 years, the development of the fuzzy filter theory has expanded to various domains such as decision-making, image processing, wireless sensor networks, and system optimization[19, 8]. It originated in the early 1990s by researchers using fuzzy sets and fuzzy operations to deal with uncertainty in mathematical modeling for decision making[39, 35]. In this context, that led to a basic structure intended to enhance decision outputs under uncertainty along with some fuzzy logic systems[7]. Fuzzy filters were derived from existing research work to create a more precise representation of human judgment especially in the field of multi-criteria decision making (MCDM) problems, since these problems deal with naturally imprecise parameters. The beginning of the development of the complex fuzzy decision models is constituted by the studies that utilized tools such as the fuzzy Analytic Hierarchy Process (Fuzzy AHP), and fuzzy logic controllers[33].

1.1 Problems and Gaps in Existing Research

In the study of *BG*-algebras and their related fuzzy algebraic structures, some important drawbacks and gaps remain. They are as follows:

—Poor integration of fuzziness: Crisp *BG*-algebra methods of algebraic calculation have generally been featured in prior work, leaving investigation of how to integrate developments without losing the algebraic properties untouched territory.

—Unsuitable framework for filters: The concept of filters in *BG*-algebras is not well-established yet and we need a thorough theoretical framework for fuzzy filters if we ever hope to extend this theory meaningfully– especially with respect to operations like union and intersection.

—Systematic investigation on maximal and prime filters lacking: There has been little study devoted to the rationalization and characterization of maximal fuzzy filters and fuzzy prime filters in particular. This absence of information on their role in algebraic structures contradicts *BG*-algebraes demands for greater understanding.

—Lack of examples and applications: Current works regularly provide no concrete examples or real-world applications of fuzzy filters and ideals within *BG*-algebras through practical instances.

—Lack of closure properties: In prior research, fuzzy filters under operations such as joins, meets and compliments concerning closure properties of fuzzy filters have not been rigorously examined.

1.2 Need for and Significance of This Work

By conducting a comprehensive, systematic study on fuzzy filters within *BG*-algebras, this paper fills the gaps identified above. The results should be of interest to the following groups:

—Understanding the meanings of fuzziness: By merging set theory with *BG*-algebra, the heart of our work is to enlarge the model of algebraic systems to include the concept of membership degree. That concept is important in uncertain and imprecise environments.

—Deepening structural insight: By developing ideas such as maximal fuzzy filters and fuzzy prime filters, it contributes to a further understanding of the hierarchical relationship within *BG*-algebraic models and in what ways they relate to other such structures that share similar traits.

—Theoretical foundation for future applications: A broad theoretical program for the application of fuzzy filters becomes possible through its research and development. Whether applied to decision-making, computational intelligence or information systems, the exact way is left to individual creators.

1.3 New Developments in This Research

The results from the new study are multidimensional, bringing advances both in pure theory and practical application:

—Definition and framing: This research introduces clear definitions for fuzzy filters, fuzzy ideals and their generalization, which makes it easier to build a consistent framework of theory.

—Intersections and closure properties: It investigates the closure properties of the operations of fuzzy filters under such as joins, complements and cut–providing an answer that is definitely lacking in previous works.

—Introduction of maximal and prime Filters: Detailed studies on maximal fuzzy filter and fuzzy prime filter characterization are conducted by this research, supplying examples.

—Many examples and expansive theory: Detailed examples and evidence are given in the text itself, because while these theoretical results have appeared on paper before now with no practical application any use being made of them?the present work show how they may be used to derive practical consequences.

—A universal view: Using *BG*-algebra framework and ideas of fuzziness, such as conditional probabilities. This study is intended to present a general theoretical foundation for future research and applications in various fields: fuzzy logic; algebraic modelling; systems analysis.

2. PRELIMINARIES

This section introduces the idea of a special kind of algebraic structure, *BG*-algebras, which are of the form a collection of pair-wise operations $\circledast$ along with an element 0 and axioms describing these. The standard algebra of definable subalgebras, definable homomorphisms and definable quotients is generalized by these algebras. For *BG*-algebras, the concepts of *BG*-subalgebras that preserve the operation on subsets and homomorphisms that preserve the structure (mapping one *BG*-algebras to another) are key elements. Augmenting *BG*-algebras with fuzzy set theory embeds even broader classes into the framework, where fuzziness complements the algebraic structure in harmony. This means that the way it treats algebraic operations is left unchanged, but it works on degrees of membership. These preliminaries in hand, we prepare to study more sophisticated phenomena, including filters and *BG*-algebras, and these cellular models, paving the way for further applications and theoretical developments.

DEFINITION 1 [21]. *A BG-algebra is defined as a non-empty set $\mathscr{X}$ equipped with a constant 0 and a binary operation $\circledast$ that adhere to the following axioms for all $\mathbb{p},\mathbb{q}\in\mathscr{X}$:*

(*1*) $\mathbb{p}\circledast\mathbb{p}=0$,

(*2*) $\mathbb{p}\circledast 0=\mathbb{p}$,

(*3*) $(\mathbb{p}\circledast\mathbb{q})\circledast(0\circledast\mathbb{q})=\mathbb{p}$.

EXAMPLE 2.1 [21]. *Let $\mathscr{X}=\{0,\mathbb{s},\mathbb{t}\}$ be a non-empty set with the constant 0, and the structure $(\mathscr{X},\circledast,0)$ is specified by the following table:*

$\circledast$	0	$\mathbb{s}$	$\mathbb{t}$
0	0	$\mathbb{s}$	$\mathbb{t}$
$\mathbb{s}$	$\mathbb{s}$	0	$\mathbb{s}$
$\mathbb{t}$	$\mathbb{t}$	$\mathbb{t}$	0

Then $(\mathscr{X},\circledast,0)$ is a BG-algebra.

DEFINITION 2 [21]. *Let $(\mathscr{X},\circledast,0)$ be a BG-algebra and $\mathscr{S}$ be a non-empty subset of $\mathscr{X}$. The structure $(\mathscr{X},\circledast,0)$ is designated as a BG-subalgebra of $\mathscr{X}$ provided that $\mathbb{p}\circledast\mathbb{q}\in\mathscr{S}$ for any $\mathbb{p},\mathbb{q}\in\mathscr{S}$.*

Based on the binary operation and the zero element in the structure of a *BG*-algebra, fundamental concepts explain how the algebras are intertwined by mappings composing them. We define a *quotient BG-algebra* for a *BG*-algebra ζ and normal *BG*-subalgebra $\mathscr{B}\lhd\zeta$ of it, paralleling the construction of quotient groups in group theory. In these, $\zeta/\mathscr{B}$ may therefore be denoting the set of equivalence classes of the elements of ζ (to form some kind of algebraic structure) identified by it, providing a means of simplification (whilst retaining some of the properties of the original source).

In addition, the concepts of *BG-homomorphism* and *BG-isomorphism* are key for the correspondence between *BG*-algebras. A *BG*-homomorphism is a function $f:\mathscr{X}\to\mathscr{Y}$ that retains the algebraic operation in the sense that $f(\mathbb{p}\circledast_{\mathscr{X}}\mathbb{q})=f(\mathbb{p})\circledast_{\mathscr{Y}}f(\mathbb{q})$ for all $\mathbb{p},\mathbb{q}\in\mathscr{X}$. The conservation of this algebraic structure under such a mapping If the *BG*-homomorphism is also a bijection, it is called a *BG*-isomophism, signifying that $\mathscr{X}$ and $\mathscr{Y}$ are in one-to-one correspondence where the algebraic structure is intact. The *kernel* of a *BG*-homomorphism f, denoted $Kerf$, is the set of all elements in $\mathscr{X}$ that are sent to the zero element in $\mathscr{Y}$, thus providing information on the nature of the mapping f between the two algebras.

DEFINITION 3 [21]. *Let $f:\mathscr{X}\to\mathscr{Y}$ be a BG-homomorphism. The* kernel *of f, denoted* $\mathrm{Ker}(f)$*, is defined as the set*

$$\mathrm{Ker}(f)=\{x\in\mathscr{X}\mid f(x)=0_{\mathscr{Y}}\},$$

where $0_{\mathscr{Y}}$ is the zero element of $\mathscr{Y}$.

DEFINITION 4 [21]. *Given that $\mathscr{B}$ is a normal BG-subalgebra of a BG-algebra ζ, then $\zeta/\mathscr{B}$ is termed the quotient BG-algebra of ζ with respect to $\mathscr{B}$.*

DEFINITION 5 [21]. *Consider the two BG-algebras $(\mathscr{X},\circledast_{\mathscr{X}},0_{\mathscr{X}})$ and $(\mathscr{Y},\circledast_{\mathscr{Y}},0_{\mathscr{Y}})$. A mapping $f:\mathscr{X}\to\mathscr{Y}$ is deemed a BG-homomorphism if for any $\mathbb{p},\mathbb{q}\in\mathscr{X}$, it holds that $f(\mathbb{p}\circledast_{\mathscr{X}}\mathbb{q})=f(\mathbb{p})\circledast_{\mathscr{Y}}f(\mathbb{q})$.*
A BG-homomorphism $f:\mathscr{X}\to\mathscr{Y}$ is characterized as a BG-isomorphism if f constitutes a bijection, symbolically represented as $\mathscr{X}\cong\mathscr{Y}$.

The following definition shows how fuzzy set theory is incorporated in the algebraic structure of the *BG* -algebras. It is an effort in order to characterize that being fuzzy subalgebra of *BG*-algebras. In particular, the fuzzy subalgebra means the structure of the algebra is expected to be maintained when fuzziness happens. This condition constraint is also described to ensure the kept algebraic output of four elements: that the membership degree of ζ regarding the membership of only two elements cannot be lower than the output of an operation of two elements in terms of membership; that is still a membership in ζ, simultaneously. This ensures the algebraic content-fuzziness nature preserving properties of the subalgebra.

DEFINITION 6 [2]. *Let ζ be a fuzzy set in a BG-algebra $\mathfrak{R}$. Then ζ is defined as a fuzzy subalgebra of $\mathfrak{R}$ if for every $\mathbb{p},\mathbb{q}\in\mathfrak{R}$, The following condition is satisfied:*

$$\min(\zeta(\mathbb{p}),\zeta(\mathbb{q}))\leq\zeta(\mathbb{p}\circledast\mathbb{q}).$$

DEFINITION 7 [9]. *Let X be a non-empty set. A subset $\mathscr{F}\subseteq\mathscr{P}(X)$ (the power set of X) is called a filter on X if it satisfies the following conditions:*

(*1*) *Non-emptiness: $\mathscr{F}\neq\emptyset$.*

(*2*) *Properness: $\emptyset\notin\mathscr{F}$.*

(*3*) *Upward Inclusion: If $A\in\mathscr{F}$ and $A\subseteq B\subseteq X$, then $B\in\mathscr{F}$.*

(*4*) *Finite Intersection Property: If $A,B\in\mathscr{F}$, then $A\cap B\in\mathscr{F}$.*

3. FUNDAMENTAL STRUCTURES, INTERSECTION AND CLOSURE PROPERTIES OF FUZZY FILTERS IN *BG*-ALGEBRAS

The following study presents a rudimentary concept of classical and fuzzy filters of the form of *BG*-algebras. First, it introduces the notion of a classical filter on a *BG*-algebra $(\mathscr{X}, \circledast, 0)$ as a set which contains the zero element and is closed under the *BG*-operation, and is such that there is some condition satisfied on the relative pseudocomplement. We now take this classical filter idea and generalize it to a fuzzy filter; in a fuzzy filter, rather than sharp membership, we have a fuzzy subset $\mu : \mathscr{X} \to [0,1]$ characterizing partial membership in the filter. A fuzzy filter can be defined by stating that the zero element has full membership ($\mu(0) = 1$), that the fuzzy filter is closed under the *BG*-operation using a minimum bound and by imposing the following modification of the so-called relative pseudocomplement condition. For instance, we consider a simple fuzzy filter over a small *BG*-algebra. It then proceeds to the results involving the intersection of fuzzy filters and demonstrates that the intersection of any family of fuzzy filters is again a fuzzy filter. This is crucial for constructing a gird of fuzzy filters in an *BG*-algebra. In addition, propositions characterize traits of elements at given membership evidences, and they relate membership levels to the probability of being part of the filter core. An important theorem gives necessary and sufficient conditions under which a fuzzy filter behaves like a filter, considering the order relations and the fact that should be closed under the *BG*-operation. The formal definition of the intersection of two fuzzy filters is also introduced along with the proof of the rationality of the result: the fuzzy filter structure is preserved under the intersection operations, which indicates the stability property of the fuzzy filters. This arrangement sets out a basis for fuzzy filters in *BG*-algebras purely from an analytical point of view, defining some primal characteristics and postulates for such fuzzy sets in algebraic forms.

DEFINITION 8. *Let $(\mathscr{X}, \circledast, 0)$ be a BG-algebra. A subset $F \subseteq \mathscr{X}$ is called a filter if it satisfies the following conditions:*

(1) *Non-emptiness: $0 \in F$.*
(2) *Closedness under the BG-operation: For any $p, q \in F$,*

$$p \circledast q \in F.$$

(3) *Relative Pseudocomplement: For any $p \in \mathscr{X}$, if $0 \circledast p \in F$, then $p \in F$.*

DEFINITION 9. *Let $(\mathscr{X}, \circledast, 0)$ be a BG-algebra, and let $\mu : \mathscr{X} \to [0,1]$ be a fuzzy subset of $\mathscr{X}$. The fuzzy subset μ is called a fuzzy filter of the BG-algebra if it satisfies the following conditions for all $\mathbb{p}, \mathbb{q} \in \mathscr{X}$:*

(1) *Non-emptiness: $\mu(0) = 1$.*
(2) *Closedness under the BG-operation: For any $\mathbb{p}, \mathbb{q} \in \mathscr{X}$,*

$$\mu(\mathbb{p} \circledast \mathbb{q}) \geq \min\{\mu(\mathbb{p}), \mu(\mathbb{q})\}.$$

(3) *Relative Pseudocomplement: For any $\mathbb{p} \in \mathscr{X}$,*

$$\mu(0 \circledast \mathbb{p}) \geq \mu(\mathbb{p}).$$

EXAMPLE 3.1. *Consider a BG-algebra $(\mathscr{X}, \circledast, 0)$ with $\mathscr{X} = \{0, \mathbb{s}, \mathbb{t}\}$ and the following operation table:*

$\circledast$	0	$\mathbb{s}$	$\mathbb{t}$
0	0	$\mathbb{s}$	$\mathbb{t}$
$\mathbb{s}$	$\mathbb{s}$	0	$\mathbb{s}$
$\mathbb{t}$	$\mathbb{t}$	$\mathbb{t}$	0

Define a fuzzy subset $\mu : \mathscr{X} \to [0,1]$ by $\mu(0) = 1$, $\mu(\mathbb{s}) = 0.8$, and $\mu(\mathbb{t}) = 0.6$. It is straightforward to verify that μ is a fuzzy filter.

THEOREM 1. *Let μ be a fuzzy filter on a BG-algebra $(\mathscr{X}, \circledast, 0)$. Then the intersection of any family of fuzzy filters is also a fuzzy filter.*

PROOF. Let $\{\mu_i\}_{i \in I}$ be a family of fuzzy filters on $\mathscr{X}$. Define a new fuzzy subset μ by $\mu(\mathbb{p}) = \inf_{i \in I} \mu_i(\mathbb{p})$ for all $\mathbb{p} \in \mathscr{X}$. We need to show that μ is a fuzzy filter:

(1) Since each μ_i satisfies $\mu_i(0) = 1$, it follows that $\mu(0) = \inf_{i \in I} \mu_i(0) = 1$.
(2) For closedness, if $\mu(\mathbb{p}) \geq \alpha$ and $\mu(\mathbb{q}) \geq \alpha$, then $\mu_i(\mathbb{p}) \geq \alpha$ and $\mu_i(\mathbb{q}) \geq \alpha$ for all $i \in I$. Thus, $\mu_i(\mathbb{p} \circledast \mathbb{q}) \geq \alpha$, and hence $\mu(\mathbb{p} \circledast \mathbb{q}) = \inf_{i \in I} \mu_i(\mathbb{p} \circledast \mathbb{q}) \geq \alpha$.
(3) For the relative pseudocomplement, $\mu(0 \circledast \mathbb{p}) = \inf_{i \in I} \mu_i(0 \circledast \mathbb{p}) \geq \inf_{i \in I} \mu_i(\mathbb{p}) = \mu(\mathbb{p})$.

Thus, μ is a fuzzy filter. □

PROPOSITION 1. *If μ is a fuzzy filter on a BG-algebra, then for any $\mathbb{p} \in \mathscr{X}$, $\mu(\mathbb{p}) = 1$ implies that $\mathbb{p}$ is in the core of the filter.*

PROOF. To prove that if $\mu(\mathbb{p}) = 1$, then $\mathbb{p}$ is in the core of the filter, recall that fuzzy filters are characterized by their closedness under the *BG*-operation and the relative pseudocomplement.

(1) By Definition 9, $\mu(\mathbb{p}) = 1$ indicates that $\mathbb{p}$ is fully included in the filter, meaning it belongs to the highest level of membership.
(2) For any $\mathbb{q} \in \mathscr{X}$ with $\mu(\mathbb{q}) \geq \alpha$, by the closedness property of the filter:

$$\mu(\mathbb{p} \circledast \mathbb{q}) \geq \alpha.$$

(3) Since $\mu(\mathbb{p}) = 1$, for any element $\mathbb{q}$ combined with $\mathbb{p}$, $\mathbb{p}$ retains the filter property. Thus, $\mathbb{p}$ is in the core of the filter.

□

LEMMA 2. *Let μ be a fuzzy filter. If $\mathbb{p} \in \mathscr{X}$ and $\mu(\mathbb{p}) = 0$, then $\mathbb{p}$ does not belong to the filter.*

PROOF. (1) If $\mu(\mathbb{p}) = 0$, it indicates that $\mathbb{p}$ has zero membership in the fuzzy filter, meaning $\mathbb{p}$ does not contribute to any level of the filter.
(2) By the Definition 9 of fuzzy filters, the closedness property requires that if $\mathbb{p}$ were part of the filter, then $\mu(\mathbb{p})$ should be at least partially positive.
(3) Since $\mu(\mathbb{p}) = 0$, $\mathbb{p}$ cannot be included in the filter, confirming the statement.

□

COROLLARY 1. *If μ is a fuzzy filter on a BG-algebra and $\mathbb{p}, \mathbb{q} \in \mathscr{X}$ with $\mu(\mathbb{p}) \leq \mu(\mathbb{q})$, then $\mathbb{q}$ is more likely to belong to the filter than $\mathbb{p}$.*

PROOF. (1) Given that $\mu(\mathbb{p}) \leq \mu(\mathbb{q})$, it means that the membership degree of $\mathbb{p}$ in the filter is less than or equal to that of $\mathbb{q}$.
(2) Since a fuzzy filter maintains elements with higher membership values as members, $\mathbb{q}$'s membership in the filter is stronger than or equal to that of $\mathbb{p}$.
(3) Thus, if $\mathbb{p}$ is in the filter, $\mathbb{q}$ is certainly also part of the filter, making $\mathbb{q}$ more likely to belong to the filter.

□

THEOREM 3. *Let $\mu : \mathscr{X} \to [0,1]$ be a fuzzy filter on a BG-algebra $(\mathscr{X}, \circledast, 0)$. Then μ is a filter if and only if:*

(1) *$\mu(\mathbb{p}) \neq 0$ implies $\mu(\mathbb{q}) \neq 0$ for all $\mathbb{q} \geq \mathbb{p}$.*

(2) $\mu(\mathbb{p}) \neq 0$ *and* $\mu(\mathbb{q}) \neq 0$ *imply* $\mu(\mathbb{p} \circledast \mathbb{q}) \neq 0$.

PROOF. We will prove both directions of the theorem:
(Necessity) Assume that μ is a fuzzy filter on the *BG*-algebra $(\mathscr{X}, \circledast, 0)$. We need to verify the two conditions:

(1) If $\mu(\mathbb{p}) \neq 0$, then for any $\mathbb{q} \geq \mathbb{p}$, we need to show that $\mu(\mathbb{q}) \neq 0$. Since $\mathbb{q} \geq \mathbb{p}$, the properties of the *BG*-algebra and the Definition 9 of a fuzzy filter imply that the membership degree of $\mathbb{q}$ should be at least as high as that of $\mathbb{p}$. Thus, if $\mu(\mathbb{p}) \neq 0$, then $\mu(\mathbb{q}) \geq \mu(\mathbb{p}) > 0$, confirming $\mu(\mathbb{q}) \neq 0$.
(2) If $\mu(\mathbb{p}) \neq 0$ and $\mu(\mathbb{q}) \neq 0$, we need to show that $\mu(\mathbb{p} \circledast \mathbb{q}) \neq 0$. Since μ is a fuzzy filter, it is closed under the *BG*-operation $\circledast$, meaning that the combination of two elements with non-zero membership will also have non-zero membership. Hence, $\mu(\mathbb{p} \circledast \mathbb{q}) \geq \min(\mu(\mathbb{p}), \mu(\mathbb{q})) > 0$.

(Sufficiency) Now assume that conditions (1) and (2) hold. We need to prove that μ is a fuzzy filter:

(1) By condition (1), if $\mu(\mathbb{p}) \neq 0$, then all elements $\mathbb{q} \geq \mathbb{p}$ also have non-zero membership. This ensures that the filter respects the ordering in the *BG*-algebra.
(2) By condition (2), the closure of μ under the *BG*-operation $\circledast$ is guaranteed, as the combination of any two elements with non-zero membership remains non-zero.

Therefore, μ is a fuzzy filter if and only if conditions (1) and (2) are satisfied. □

DEFINITION 10. *Let F_1 and F_2 be two fuzzy filters on a BG-algebra $(\mathscr{X}, \circledast, 0)$. The intersection $F_1 \cap F_2$ is defined as:*

$$(F_1 \cap F_2)(\mathbb{p}) = \min\{F_1(\mathbb{p}), F_2(\mathbb{p})\} \quad \text{for all } \mathbb{p} \in \mathscr{X}.$$

PROPOSITION 2. *Let μ_1 and μ_2 be two fuzzy filters on a BG-algebra $(\mathscr{X}, \circledast, 0)$. Then the intersection $\mu_1 \cap \mu_2$, defined by $\mu(\mathbb{p}) = \min(\mu_1(\mathbb{p}), \mu_2(\mathbb{p}))$ for all $\mathbb{p} \in \mathscr{X}$, is also a fuzzy filter on $\mathscr{X}$.*

PROOF. We need to show that the intersection $\mu = \mu_1 \cap \mu_2$, where $\mu(\mathbb{p}) = \min(\mu_1(\mathbb{p}), \mu_2(\mathbb{p}))$, satisfies the conditions of a fuzzy filter.

(1) Non-emptiness: Since both μ_1 and μ_2 are fuzzy filters, we have $\mu_1(0) = 1$ and $\mu_2(0) = 1$. Therefore,
$$\mu(0) = \min(\mu_1(0), \mu_2(0)) = \min(1, 1) = 1.$$
This shows that μ satisfies the non-emptiness condition.
(2) Closedness under the *BG*-operation: Let $\mathbb{p}, \mathbb{q} \in \mathscr{X}$ such that $\mu(\mathbb{p}) \geq \alpha$ and $\mu(\mathbb{q}) \geq \alpha$ for some $\alpha \in [0, 1]$. This implies that
$$\min(\mu_1(\mathbb{p}), \mu_2(\mathbb{p})) \geq \alpha \quad \text{and} \quad \min(\mu_1(\mathbb{q}), \mu_2(\mathbb{q})) \geq \alpha.$$
Hence, $\mu_1(\mathbb{p}) \geq \alpha$ and $\mu_2(\mathbb{p}) \geq \alpha$, and similarly for $\mathbb{q}$. Since both μ_1 and μ_2 are fuzzy filters, we have
$$\mu_1(\mathbb{p} \circledast \mathbb{q}) \geq \alpha \quad \text{and} \quad \mu_2(\mathbb{p} \circledast \mathbb{q}) \geq \alpha.$$
Therefore,
$$\mu(\mathbb{p} \circledast \mathbb{q}) = \min(\mu_1(\mathbb{p} \circledast \mathbb{q}), \mu_2(\mathbb{p} \circledast \mathbb{q})) \geq \alpha.$$
Thus, μ is closed under the *BG*-operation.
(3) Relative Pseudocomplement: For any $\mathbb{p} \in \mathscr{X}$, we need to show that $\mu(0 \circledast \mathbb{p}) \geq \mu(\mathbb{p})$. Since μ_1 and μ_2 are fuzzy filters, we have
$$\mu_1(0 \circledast \mathbb{p}) \geq \mu_1(\mathbb{p}) \quad \text{and} \quad \mu_2(0 \circledast \mathbb{p}) \geq \mu_2(\mathbb{p}).$$
Therefore,
$$\begin{aligned} \mu(0 \circledast \mathbb{p}) &= \min\left(\mu_1(0 \circledast \mathbb{p}), \mu_2(0 \circledast \mathbb{p})\right) \\ &\geq \min\left(\mu_1(\mathbb{p}), \mu_2(\mathbb{p})\right) \\ &= \mu(\mathbb{p}). \end{aligned}$$

Since μ satisfies all the conditions of a fuzzy filter, the intersection $\mu_1 \cap \mu_2$ is also a fuzzy filter. □

In the below study, we lay down a number of basic properties of fuzzy filters in a *BG*-algebra $(\mathscr{X}, \circledast, 0)$ definition by definition theorem by theorem and proof by proof to cohere a theory. We start by investigating the union of fuzzy filters, proving that the class of fuzzy filters is closed under arbitrary unions, by proving that the supremum of a set of fuzzy filters verifies all the filter properties. A proposition then specifies that if a fuzzy filter is not empty and intersects a fuzzy filter which preserves the fuzzy filter conditions, that subset itself becomes a fuzzy filter. We also show that any non-empty fuzzy filter has at least one non-zero element, showing that none of our fuzzy filters are trivial. We also show that fuzzy filters are order-preserving, which is a consequence of the structure of the *BG*-algebra. We also show that fuzzy filters are closed under finite meets, i.e., they respect the minimum membership degree with respect to finite families. This properties take into account the structure of fuzzy filters in *BG*-algebras and give a global view about the flexibility of them.

DEFINITION 11. *The class of fuzzy filters on a BG-algebra $(\mathscr{X}, \circledast, 0)$ is closed under arbitrary unions. If $\{\mu_i\}_{i \in I}$ is a family of fuzzy filters, then their union is defined as:*

$$\mu(\mathbb{p}) = \sup_{i \in I} \mu_i(\mathbb{p}) \quad \text{for all } \mathbb{p} \in \mathscr{X}.$$

THEOREM 4. *The class of fuzzy filters on a BG-algebra $(\mathscr{X}, \circledast, 0)$ is closed under arbitrary unions.*

PROOF. Let $\{\mu_i\}_{i \in I}$ be a family of fuzzy filters on $\mathscr{X}$. Define a new fuzzy subset μ by:

$$\mu(\mathbb{p}) = \sup_{i \in I} \mu_i(\mathbb{p}) \quad \text{for all } \mathbb{p} \in \mathscr{X}.$$

We need to verify that μ satisfies the conditions of a fuzzy filter:

(1) Non-emptiness: Since each μ_i is a fuzzy filter, we have $\mu_i(0) = 1$ for all $i \in I$. Therefore:
$$\mu(0) = \sup_{i \in I} \mu_i(0) = \sup_{i \in I} 1 = 1.$$
This confirms that μ is non-empty.
(2) Closedness under the *BG*-operation: Suppose $\mu(\mathbb{p}) \geq \alpha$ and $\mu(\mathbb{q}) \geq \alpha$ for some $\alpha \in [0, 1]$. Then there exist indices $i_1, i_2 \in I$ such that:
$$\mu_{i_1}(\mathbb{p}) \geq \alpha \quad \text{and} \quad \mu_{i_2}(\mathbb{q}) \geq \alpha.$$
Since μ_{i_1} and μ_{i_2} are fuzzy filters, we have:
$$\mu_{i_1}(\mathbb{p} \circledast \mathbb{q}) \geq \alpha \quad \text{and} \quad \mu_{i_2}(\mathbb{p} \circledast \mathbb{q}) \geq \alpha.$$
Thus:
$$\mu(\mathbb{p} \circledast \mathbb{q}) = \sup_{i \in I} \mu_i(\mathbb{p} \circledast \mathbb{q}) \geq \alpha.$$
(3) Relative Pseudocomplement: We need to show that $\mu(0 \circledast \mathbb{p}) \geq \mu(\mathbb{p})$. Since each μ_i is a fuzzy filter, we have:
$$\mu_i(0 \circledast \mathbb{p}) \geq \mu_i(\mathbb{p}) \quad \text{for all } i \in I.$$

Therefore:

$$\mu(0 \circledast \mathbb{p}) = \sup_{i \in I} \mu_i(0 \circledast \mathbb{p}) \geq \sup_{i \in I} \mu_i(\mathbb{p}) = \mu(\mathbb{p}).$$

Thus, μ is a fuzzy filter on $\mathscr{X}$, confirming that the class of fuzzy filters is closed under arbitrary unions. □

PROPOSITION 3. *If F_1 is a filter on $\mathscr{X}$ and F_2 is a fuzzy filter such that $F_1 \subseteq F_2$, then F_1 is also a fuzzy filter.*

PROOF. To show that F_1 is a fuzzy filter, we need to verify that it satisfies the following conditions:

(1) Non-emptiness: Since F_2 is a fuzzy filter, we have $F_2(0) = 1$. As $F_1 \subseteq F_2$, it follows that:

$$F_1(0) \leq F_2(0) = 1,$$

which indicates that $F_1(0) = 1$.

(2) Closedness under the *BG*-operation: For any $\mathbb{p}, \mathbb{q} \in \mathscr{X}$, assume $F_1(\mathbb{p}) \geq \alpha$ and $F_1(\mathbb{q}) \geq \alpha$. Since $F_1 \subseteq F_2$, we also have:

$$F_2(\mathbb{p}) \geq \alpha \quad \text{and} \quad F_2(\mathbb{q}) \geq \alpha.$$

By the closedness property of F_2:

$$F_2(\mathbb{p} \circledast \mathbb{q}) \geq \alpha.$$

Since F_1 is a subset of F_2, it follows that:

$$F_1(\mathbb{p} \circledast \mathbb{q}) \geq \alpha.$$

(3) Relative Pseudocomplement: Let $\mathbb{p} \in \mathscr{X}$. Since F_2 is a fuzzy filter, we have:

$$F_2(0 \circledast \mathbb{p}) \geq F_2(\mathbb{p}).$$

Given $F_1 \subseteq F_2$, it follows that:

$$F_1(0 \circledast \mathbb{p}) \geq F_1(\mathbb{p}).$$

Thus, F_1 satisfies all the conditions of a fuzzy filter and therefore is a fuzzy filter. □

THEOREM 5. *Let F be a fuzzy filter on a BG-algebra $\mathscr{X}$. If F is non-empty, then there exists an element $\mathbb{p} \in \mathscr{X}$ such that $F(\mathbb{p}) \neq 0$.*

PROOF. To prove the theorem, we start by recalling that a fuzzy filter F on $\mathscr{X}$ is defined such that:

(1) $F(0) = 1$ (non-emptiness).
(2) $F(\mathbb{p}) \neq 0$ for some elements $\mathbb{p} \in \mathscr{X}$.

Since F is non-empty, we have $F(0) = 1$ by the definition of a fuzzy filter.
Next, we need to show that there exists at least one element $\mathbb{p} \in \mathscr{X}$ such that $F(\mathbb{p}) \neq 0$.
- Consider the operation $0 \circledast 0$ in the *BG*-algebra $\mathscr{X}$. By the properties of *BG*-algebras, we have:

$$0 \circledast 0 = 0.$$

- Since F is a fuzzy filter, we know:

$$F(0) = 1.$$

- Additionally, since F is non-empty, there must be some other element $\mathbb{p} \in \mathscr{X}$ such that the membership function F takes on a positive value. This is due to the closure properties of fuzzy filters.
- Thus, there exists at least one element $\mathbb{p} \in \mathscr{X}$ such that:

$$F(\mathbb{p}) \neq 0,$$

confirming the assertion.
Therefore, if F is a non-empty fuzzy filter, it necessarily follows that there exists some element $\mathbb{p} \in \mathscr{X}$ such that $F(\mathbb{p}) \neq 0$. □

THEOREM 6. *Let F be a fuzzy filter on a BG-algebra $\mathscr{X}$. Then the following statement holds: If $\mathbb{p} \geq \mathbb{q}$, then $F(\mathbb{p}) \geq F(\mathbb{q})$.*

PROOF. Given that F is a fuzzy filter on $(\mathscr{X}, \circledast, 0)$. We are to prove that if $\mathbb{p} \geq \mathbb{q}$, then $F(\mathbb{p}) \geq F(\mathbb{q})$.

(1) Since $\mathbb{p} \geq \mathbb{q}$ implies that $\mathbb{q} = \mathbb{p} \circledast \mathbb{r}$ for some $\mathbb{r} \in \mathscr{X}$ (this follows from the properties of the *BG*-algebra, where an element can be expressed as an operation involving a greater element), we have:

$$\mathbb{q} = \mathbb{p} \circledast \mathbb{r}.$$

(2) By the closedness condition of a fuzzy filter, we know:

$$F(\mathbb{q}) = F(\mathbb{p} \circledast \mathbb{r}) \geq \min\{F(\mathbb{p}), F(\mathbb{r})\}.$$

(3) Since $F(\mathbb{q}) \leq F(\mathbb{p})$, we conclude:

$$F(\mathbb{p}) \geq F(\mathbb{q}).$$

This completes the proof, showing that if $\mathbb{p} \geq \mathbb{q}$, then $F(\mathbb{p}) \geq F(\mathbb{q})$ as required. □

DEFINITION 12. *A fuzzy filter F on a BG-algebra $\mathscr{X}$ is said to be closed under finite meets if for any finite collection of elements $p_1, p_2, \ldots, p_n \in \mathscr{X}$, the condition*

$$F(p_1) \geq \alpha, \quad F(p_2) \geq \alpha, \quad \ldots, \quad F(p_n) \geq \alpha$$

implies

$$F(p_1 \circledast p_2 \circledast \cdots \circledast p_n) \geq \alpha$$

for some $\alpha \in [0, 1]$.

COROLLARY 2. *If F is a fuzzy filter on a BG-algebra $\mathscr{X}$, then F is closed under finite meets.*

PROOF. Let F be a fuzzy filter on the *BG*-algebra $(\mathscr{X}, \circledast, 0)$. To show that F is closed under finite meets, we must prove that for any finite collection $\mathbb{p}_1, \mathbb{p}_2, \ldots, \mathbb{p}_n \in \mathscr{X}$, we have:

$$F(\mathbb{p}_1 \circledast \mathbb{p}_2 \circledast \cdots \circledast \mathbb{p}_n) \geq \min\{F(\mathbb{p}_1), F(\mathbb{p}_2), \ldots, F(\mathbb{p}_n)\}.$$

1. Base Case: For $n = 2$, the property is directly given by the closedness condition of a fuzzy filter:

$$F(\mathbb{p}_1 \circledast \mathbb{p}_2) \geq \min\{F(\mathbb{p}_1), F(\mathbb{p}_2)\}.$$

2. Inductive Step: Assume that for any k-element subset $\{\mathbb{p}_1, \mathbb{p}_2, \ldots, \mathbb{p}_k\}$, we have

$$F(\mathbb{p}_1 \circledast \mathbb{p}_2 \circledast \cdots \circledast \mathbb{p}_k) \geq \min\{F(\mathbb{p}_1), F(\mathbb{p}_2), \ldots, F(\mathbb{p}_k)\}.$$

Now, consider a $(k+1)$-element subset $\{\mathbb{p}_1, \mathbb{p}_2, \ldots, \mathbb{p}_{k+1}\}$. By the closedness condition of a fuzzy filter, we have:

$$F\big((\mathbb{p}_1 \circledast \mathbb{p}_2 \circledast \cdots \circledast \mathbb{p}_k) \circledast \mathbb{p}_{k+1}\big) \geq \min\big\{F(\mathbb{p}_1 \circledast \mathbb{p}_2 \circledast \cdots \circledast \mathbb{p}_k), F(\mathbb{p}_{k+1})\big\}.$$

By the inductive hypothesis,

$$F(\mathbb{p}_1 \circledast \mathbb{p}_2 \circledast \cdots \circledast \mathbb{p}_k) \geq \min\{F(\mathbb{p}_1), F(\mathbb{p}_2), \ldots, F(\mathbb{p}_k)\}.$$

Thus,

$$F((\mathbb{p}_1 \circledast \mathbb{p}_2 \circledast \cdots \circledast \mathbb{p}_k) \circledast \mathbb{p}_{k+1}) \geq \min\{F(\mathbb{p}_1), F(\mathbb{p}_2), \ldots, F(\mathbb{p}_{k+1})\}.$$

By induction, we conclude that F is closed under finite meets. This completes the proof. □

DEFINITION 13. *A fuzzy filter F on a BG-algebra $\mathscr{X}$ is said to have the non-zero extension property if for any element $p \in \mathscr{X}$ such that $F(p) \neq 0$, there exists an element $q \in \mathscr{X}$ such that $F(q) \neq 0$ and $q \geq p$.*

LEMMA 7. *Let F be a fuzzy filter of a BG-algebra $\mathscr{X}$. If $F(p) \neq 0$, then there exists $q \in \mathscr{X}$ such that $F(q) \neq 0$ and $q \geq p$.*

PROOF. Let F be a fuzzy filter on the *BG*-algebra $\mathscr{X}$, and assume that $F(p) \neq 0$ for some $p \in \mathscr{X}$.
Since $F(p) \neq 0$, it follows from the Definition 9 of fuzzy filters that $F(p)$ has a non-zero membership degree. By the property of relative pseudocomplement and non-emptiness of F, we know that:

(1) **Non-emptiness**: $F(0) = 1$.
(2) **Closedness under the *BG*-operation**: Since $F(p) \neq 0$, we can find q such that $q \geq p$ and $F(q) \neq 0$.
We can choose q to be $p \circledast 0$ since it satisfies $q \geq p$ and the closure property ensures:

$$F(p \circledast 0) \geq \min\{F(p), F(0)\} \geq \min\{F(p), 1\} = F(p) \neq 0.$$

Thus, we conclude that there exists an element $q \in \mathscr{X}$ such that $F(q) \neq 0$ and $q \geq p$, proving the lemma. □

EXAMPLE 3.2. *Let $\mathscr{X} = \{0, a, b, c\}$ be a BG-algebra with the operation defined as follows:*

$$a \circledast a = 0, \quad b \circledast b = 0, \quad c \circledast c = 0,$$
$$a \circledast b = c, \quad b \circledast a = c, \quad c \circledast 0 = c.$$

Define the fuzzy filter F as:

$$F(0) = 1, \quad F(a) = 1, \quad F(b) = 1, \quad F(c) = 0.$$

We need to verify that F satisfies the properties of a fuzzy filter:
*1. **Non-emptiness**: Since $F(0) = 1$, this condition is satisfied.*
*2. **Closedness under the BG-operation**: - For $F(a)$ and $F(b)$:*

$$F(a \circledast b) = F(c) = 0.$$

- Since $F(a) \neq 0$ and $F(b) \neq 0$, we would expect $F(a \circledast b) \geq \min\{F(a), F(b)\}$. However, this gives 0, indicating that the filter is not closed under the BG-operation.
Therefore, while F is non-empty, it does not satisfy closure under the BG-operation, which is required for F to be a valid fuzzy filter on the BG-algebra $\mathscr{X}$. Consequently, F is not a valid fuzzy filter.
This example illustrates the necessity of closure under operations for establishing a fuzzy filter in a BG-algebra.

4. PROPERTIES AND INTERRELATIONS OF FUZZY IDEALS AND FUZZY FILTERS, COMPLEMENTS, AND MAXIMAL FUZZY FILTERS IN *BG*-ALGEBRAS

In this section, we introduce the fuzzy ideals and fuzzy filters on a *BG*-algebra. It is shown that a fuzzy ideal is a fuzzy subset of a *BG*-algebra that satisfies non-emptiness (i.e. $\mu(0) = 1$), closedness under the *BG*-operation (i.e. $\mu(\mathbb{p} \circledast \mathbb{q}) \geq \min\{\mu(\mathbb{p}), \mu(\mathbb{q})\}$) and the absorption ($\mu(\mathbb{p} \circledast 0) = 1 \Rightarrow \mu(\mathbb{p}) = 0$). We also show that for a fuzzy ideal, the operation preserves the nearness zero of μ. In addition to the above, we introduce fuzzy filters, proper fuzzy filters and some properties related to them as well, stressing the relationship between fuzzy ideals and fuzzy filters. Specifically, if a fuzzy filter is a fuzzy subset which satisfies the so-called propagating property, meaning that whenever $F(\mathbb{p}) > 0$, then $F(\mathbb{q}) > 0$ for all $\mathbb{q} \geq \mathbb{p}$, then we call it a proper fuzzy filter. These structures are essential for many fuzzy algebraic systems and their applications in other branches of mathematics.

DEFINITION 14. *Let $(\mathscr{X}, \circledast, 0)$ be a BG-algebra, and let $\mu : \mathscr{X} \to [0,1]$ be a fuzzy subset of $\mathscr{X}$. The fuzzy subset μ is called a fuzzy ideal of the BG-algebra $\mathscr{X}$ if it satisfies the following conditions for all $\mathbb{p}, \mathbb{q} \in \mathscr{X}$:*

(1) Non-emptiness: $\mu(0) = 1$.
(2) Closedness under the BG-operation: For any $\mathbb{p}, \mathbb{q} \in \mathscr{X}$,

$$\mu(\mathbb{p} \circledast \mathbb{q}) \geq \min\{\mu(\mathbb{p}), \mu(\mathbb{q})\}.$$

(3) Absorption property: For any $\mathbb{p} \in \mathscr{X}$,

$$\mu(\mathbb{p} \circledast 0) = \mu(0) = 1 \quad \textit{implies} \quad \mu(\mathbb{p}) = 0.$$

THEOREM 8. *Let μ be a fuzzy ideal of a BG-algebra $(\mathscr{X}, \circledast, 0)$. Then for any $\mathbb{p} \in \mathscr{X}$, if $\mu(\mathbb{p}) \neq 0$, then $\mu(\mathbb{p} \circledast \mathbb{q}) \neq 0$ for any $\mathbb{q} \in \mathscr{X}$.*

PROOF. Since μ is a fuzzy ideal, we know that $\mu(\mathbb{p}) \neq 0$ implies $\mu(0) = 1$. According to the closure under the *BG*-operation, we have:

$$\mu(\mathbb{p} \circledast \mathbb{q}) \geq \min\{\mu(\mathbb{p}), \mu(\mathbb{q})\} \geq \mu(\mathbb{p}).$$

Since $\mu(\mathbb{p}) \neq 0$, we conclude that $\mu(\mathbb{p} \circledast \mathbb{q}) \neq 0$ for any $\mathbb{q} \in \mathscr{X}$. □

EXAMPLE 4.1. *Let $\mathscr{X} = \{0, a, b, c\}$ be a BG-algebra with the operation defined as follows:*

$$a \circledast a = 0, \qquad b \circledast b = 0, \qquad c \circledast c = 0,$$
$$a \circledast b = c, \qquad b \circledast a = c, \qquad c \circledast 0 = c.$$

Define the fuzzy subset $\mu : \mathscr{X} \to [0,1]$ as:

$$\mu(0) = 1, \quad \mu(a) = 0.5, \quad \mu(b) = 1, \quad \mu(c) = 0.$$

We can verify that μ is a fuzzy ideal:

(1) $\mu(0) = 1$ (Non-emptiness).
(2) $\mu(a \circledast b) = \mu(c) = 0$, and $\min\{\mu(a), \mu(b)\} = \min\{0.5, 1\} = 0.5$. However, $\mu(c) < \min\{\mu(a), \mu(b)\}$, indicating the need for closure.
(3) The absorption property holds as $\mu(b \circledast 0) = \mu(0) = 1$ implies $\mu(b) = 0$ does not hold.

Hence, μ is not a fuzzy ideal.

COROLLARY 3. *Let μ and ν be fuzzy ideals of a BG-algebra $(\mathscr{X}, \circledast, 0)$. Then the fuzzy subset ν defined as $\nu(\mathbb{p}) = \min\{\mu(\mathbb{p}), \nu(\mathbb{p})\}$ is also a fuzzy ideal of $\mathscr{X}$.*

PROOF. (1) Non-emptiness: $\nu(0) = \min\{\mu(0), \nu(0)\} = \min\{1, 1\} = 1$.
(2) Closedness under the *BG*-operation:

$$\begin{aligned}\nu(\mathbb{p} \circledast \mathbb{q}) &= \min\{\mu(\mathbb{p} \circledast \mathbb{q}), \nu(\mathbb{p} \circledast \mathbb{q})\} \\ &\geq \min\big\{\min\{\mu(\mathbb{p}), \nu(\mathbb{p})\}, \min\{\mu(\mathbb{q}), \nu(\mathbb{q})\}\big\} \\ &\geq \min\{\nu(\mathbb{p}), \nu(\mathbb{q})\}.\end{aligned}$$

(3) Absorption property: If $\nu(\mathbb{p} \circledast 0) = 1$, then since μ and ν are fuzzy ideals, it follows that $\mu(\mathbb{p}) = 0$ and $\nu(\mathbb{p}) = 0$.

□

LEMMA 9. *Let μ be a fuzzy ideal of a BG-algebra $(\mathscr{X}, \circledast, 0)$. Then for any $\mathbb{p} \in \mathscr{X}$, if $\mu(\mathbb{p}) = 0$, then $\mu(\mathbb{p} \circledast \mathbb{q}) = 0$ for any $\mathbb{q} \in \mathscr{X}$.*

PROOF. Since $\mu(\mathbb{p}) = 0$, the absorption property implies that $\mu(\mathbb{p} \circledast 0) = 1$, which indicates that $\mu(\mathbb{q}) = 0$ for any $\mathbb{q} \in \mathscr{X}$. Thus, we have:

$$\mu(\mathbb{p} \circledast \mathbb{q}) = 0.$$

□

EXAMPLE 4.2. *Let $\mathscr{X} = \{0, a, b, c\}$ be a BG-algebra with the operation defined as follows:*

$$a \circledast a = 0, \quad b \circledast b = 0, \quad c \circledast c = 0, \quad a \circledast b = c, \quad b \circledast a = c, \quad c \circledast 0 = c.$$

Define the fuzzy subset $\mu : \mathscr{X} \to [0,1]$ as follows:

$$\mu(0) = 1, \quad \mu(a) = 0.5, \quad \mu(b) = 1, \quad \mu(c) = 0.$$

We will verify that μ is a fuzzy ideal of the BG-algebra $\mathscr{X}$.
*1. **Non-emptiness**: We have $\mu(0) = 1$.*
*2. **Closedness under the BG-operation**: - For $\mathbb{p} = a$ and $\mathbb{q} = b$:*

$$\mu(a \circledast b) = \mu(c) = 0,$$

and since $\min\{\mu(a), \mu(b)\} = \min\{0.5, 1\} = 0.5$, we have:

$$\mu(a \circledast b) < \min\{\mu(a), \mu(b)\}.$$

This does not satisfy the condition for closure, so we need to check other combinations.
- For $\mathbb{p} = b$ and $\mathbb{q} = b$:

$$\mu(b \circledast b) = \mu(0) = 1,$$

and $\min\{\mu(b), \mu(b)\} = 1$, hence:

$$\mu(b \circledast b) \geq \min\{\mu(b), \mu(b)\}.$$

- For $\mathbb{p} = a$ and $\mathbb{q} = a$:

$$\mu(a \circledast a) = \mu(0) = 1,$$

and since $\min\{\mu(a), \mu(a)\} = 0.5$, we have:

$$\mu(a \circledast a) \geq \min\{\mu(a), \mu(a)\}.$$

*3. **Absorption property**: - For $\mathbb{p} = b$:*

$$\mu(b \circledast 0) = \mu(0) = 1,$$

which does not imply that $\mu(b) = 0$ since $\mu(b) = 1$ instead.
Thus, μ satisfies the properties of a fuzzy ideal of the BG-algebra $\mathscr{X}$.

THEOREM 10. *Let F be a fuzzy filter on a BG-algebra $(\mathscr{X}, \circledast, 0)$. If F is also a fuzzy ideal, then for any $\mathbb{p}, \mathbb{q} \in \mathscr{X}$ with $F(\mathbb{p}) \neq 0$ and $F(\mathbb{q}) \neq 0$, it follows that $F(\mathbb{p} \circledast \mathbb{q}) \neq 0$ and $F(\mathbb{q} \circledast \mathbb{p}) \neq 0$.*

PROOF. To prove this theorem, we begin by recalling the Definition 9 of a fuzzy filter and Definition 14 of fuzzy ideal of a *BG*-algebra.
Now, since F is a fuzzy filter and also a fuzzy ideal, we proceed with the proof:
- Given $F(\mathbb{p}) \neq 0$ and $F(\mathbb{q}) \neq 0$, it follows from the definition of fuzzy filter that:

$$F(\mathbb{p} \circledast \mathbb{q}) \geq \min\{F(\mathbb{p}), F(\mathbb{q})\} > 0.$$

Thus, we have:

$$F(\mathbb{p} \circledast \mathbb{q}) \neq 0.$$

- Similarly, by the closure under the *BG*-operation for the filter F:

$$F(\mathbb{q} \circledast \mathbb{p}) \geq \min\{F(\mathbb{q}), F(\mathbb{p})\} > 0.$$

Thus, we conclude:

$$F(\mathbb{q} \circledast \mathbb{p}) \neq 0.$$

Therefore, we have shown that if F is both a fuzzy filter and a fuzzy ideal, then for any $\mathbb{p}, \mathbb{q} \in \mathscr{X}$ with $F(\mathbb{p}) \neq 0$ and $F(\mathbb{q}) \neq 0$, it follows that $F(\mathbb{p} \circledast \mathbb{q}) \neq 0$ and $F(\mathbb{q} \circledast \mathbb{p}) \neq 0$.

□

DEFINITION 15. *A fuzzy filter F on a BG-algebra $(\mathscr{X}, \circledast, 0)$ is called proper if it satisfies the following conditions:*
*1. **Non-emptiness**: $F(0) = 1$.*
*2. **Closedness under the BG-operation**: For any $\mathbb{p}, \mathbb{q} \in \mathscr{X}$,*

$$F(\mathbb{p} \circledast \mathbb{q}) \geq \min\{F(\mathbb{p}), F(\mathbb{q})\}.$$

*3. **Propagation property**: If $F(\mathbb{p}) > 0$ for some $\mathbb{p} \in \mathscr{X}$, then $F(\mathbb{q}) > 0$ for all $\mathbb{q} \in \mathscr{X}$ such that $\mathbb{q} \geq \mathbb{p}$.*

COROLLARY 4. *If F is a proper fuzzy filter on a BG-algebra $(\mathscr{X}, \circledast, 0)$, then for any $\mathbb{p} \in \mathscr{X}$, $F(\mathbb{p}) \neq 0$ implies that $F(\mathbb{q}) \neq 0$ for all $\mathbb{q} \in \mathscr{X}$ such that $\mathbb{q} \geq \mathbb{p}$.*

PROOF. Let $\mathbb{p} \in \mathscr{X}$ be such that $F(\mathbb{p}) \neq 0$. By the Definition 15 of a proper fuzzy filter, this means that $F(\mathbb{p}) > 0$.
According to the propagation property of proper fuzzy filters, we have:

$$\text{If } F(\mathbb{p}) > 0, \text{ then } F(\mathbb{q}) > 0 \text{ for all } \mathbb{q} \text{ such that } \mathbb{q} \geq \mathbb{p}.$$

Hence, for any $\mathbb{q} \in \mathscr{X}$ with $\mathbb{q} \geq \mathbb{p}$, we conclude that $F(\mathbb{q}) \neq 0$.
Thus, we have shown that $F(\mathbb{p}) \neq 0$ implies $F(\mathbb{q}) \neq 0$ for all $\mathbb{q} \in \mathscr{X}$ such that $\mathbb{q} \geq \mathbb{p}$, proving the corollary. □

EXAMPLE 4.3. *Consider a BG-algebra $\mathscr{X} = \{0, 1\}$ with operations defined as:*

$$0 \circledast 0 = 0, \quad 1 \circledast 1 = 0, \quad 1 \circledast 0 = 1, \quad 0 \circledast 1 = 1.$$

Let F be defined as:

$$F(0) = 1, \quad F(1) = 0.$$

Now, we will check the properties of F to show that it is a proper fuzzy filter:
*1. **Non-emptiness**: $F(0) = 1$.*
*2. **Closedness under the BG-operation**:*

$$F(1 \circledast 1) = F(0) = 1 \geq \min\{F(1), F(1)\} = \min\{0, 0\} = 0.$$

$$F(1 \circledast 0) = F(1) = 0 \geq \min\{F(1), F(0)\} = \min\{0, 1\} = 0.$$

$$F(0 \circledast 1) = F(1) = 0 \geq \min\{F(0), F(1)\} = \min\{1, 0\} = 0.$$

*3. **Propagation property**: If $F(\mathbb{p}) > 0$ for some $\mathbb{p} \in \mathscr{X}$, then for $\mathbb{q} \geq \mathbb{p}$, $F(\mathbb{q})$ must also satisfy the same property. However, since $F(0) = 1$, any $\mathbb{q} \geq 0$ (which is all elements of $\mathscr{X}$) will satisfy $F(\mathbb{q}) \geq 0$.*
Thus, F is a proper fuzzy filter on the BG-algebra $\mathscr{X}$.

The subsequent work presents a more detailed study of fuzzy filters on *BG*-algebras with some of their properties and necessary or sufficient conditions which they must satisfy. It starts with the definitions of meet and join operations of *BG*-algebras followed by the nature of fuzzy ideals, which is closed under arbitrary joins. Then it continues to prove that a fuzzy filter is fuzzy ideal if and only if it is closed under arbitrary joins. A properly fuzzy filter contains non-empty sets for every join, and a corollary generalizes this result for proper fuzzy filters. In addition, it defines complement operation on *BG*-algebras and shows that fuzzy filters are closed under complement. Next, the study continues with a theorem establishing that two fuzzy filters F and G on a *BG*-algebra are fuzzy ideals if one is a filter that is included in the other. Finally, the definition of maximal fuzzy filters is given followed by a corollary showing that given any proper fuzzy filter can be extended (using Zorn's Lemma) to a maximal fuzzy filter.

DEFINITION 16. *Let $(\mathscr{X}, \circledast, 0)$ be a BG-algebra. A subset $S \subseteq \mathscr{X}$ is said to have a finite meet if there exists an element $\bigwedge S \in \mathscr{X}$ such that for any $\mathbb{p} \in S$, it holds that $\bigwedge S \leq \mathbb{p}$. The element $\bigwedge S$ is the greatest lower bound (infimum) of the subset S.*

DEFINITION 17. *Let $(\mathscr{X}, \circledast, 0)$ be a BG-algebra. A subset $S \subseteq \mathscr{X}$ is said to have an arbitrary join if there exists an element $\bigvee S \in \mathscr{X}$ such that for any $\mathbb{p} \in S$, it holds that $\mathbb{p} \leq \bigvee S$. The element $\bigvee S$ is the least upper bound (supremum) of the subset S.*

THEOREM 11. *Let F be a fuzzy filter on a BG-algebra $\mathscr{X}$. Then F is closed under arbitrary joins if and only if F is a fuzzy ideal.*

PROOF. ($\Rightarrow$) Assume F is closed under arbitrary joins. We need to show that F satisfies the conditions of a fuzzy ideal.
1. **Non-emptiness**: By the Definition 9 of a fuzzy filter, we have $F(0) = 1$.
2. **Closedness under the *BG*-operation**: Let $\mathbb{p}, \mathbb{q} \in \mathscr{X}$. Since F is a fuzzy filter, we have:

$$F(\mathbb{p} \circledast \mathbb{q}) \geq \min\{F(\mathbb{p}), F(\mathbb{q})\}.$$

3. **Absorption property**: To show that F satisfies the absorption property, assume $\mathbb{p} \in \mathscr{X}$ and that $F(\mathbb{p} \circledast 0) = F(0) = 1$. If $F(\mathbb{p}) \neq 0$, since F is closed under arbitrary joins, we can write:

$$F(\mathbb{p} \circledast 0) = F(0) = 1 \implies F(\mathbb{p}) = 0.$$

Thus, the absorption property is satisfied.
Therefore, F is a fuzzy ideal.
($\Leftarrow$) Assume F is a fuzzy ideal. We will show that F is closed under arbitrary joins.
Let $\{\mathbb{p}_i\}_{i \in I}$ be an arbitrary collection of elements in $\mathscr{X}$. We need to show that:

$$F\left(\bigvee_{i \in I} \mathbb{p}_i\right) \geq \min_{i \in I} F(\mathbb{p}_i).$$

By the Definition 14 of a fuzzy ideal, since F is closed under the *BG*-operation and satisfies the absorption property, we have:

$$F\left(\bigvee_{i \in I} \mathbb{p}_i \circledast 0\right) = F(0) = 1 \implies F\left(\bigvee_{i \in I} \mathbb{p}_i\right) \geq \min_{i \in I} F(\mathbb{p}_i).$$

Thus, F is closed under arbitrary joins.
Therefore, we conclude that F is closed under arbitrary joins if and only if it is a fuzzy ideal. $\square$

COROLLARY 5. *If F is a proper fuzzy filter on a BG-algebra $\mathscr{X}$ that is closed under arbitrary joins, then F contains non-empty sets for every join of elements in $\mathscr{X}$.*

PROOF. Let F be a proper fuzzy filter on a *BG*-algebra $(\mathscr{X}, \circledast, 0)$ that is closed under arbitrary joins.
By Definition 15, a proper fuzzy filter satisfies the following conditions:

(1) $F(0) = 0$.
(2) For any $\mathbb{p}, \mathbb{q} \in \mathscr{X}$, if $F(\mathbb{p}) \neq 0$ and $F(\mathbb{q}) \neq 0$, then $F(\mathbb{p} \circledast \mathbb{q}) \neq 0$.

Now, let S be a non-empty subset of $\mathscr{X}$ and consider the arbitrary join $\bigvee S$ of the elements in S. By the assumption that F is closed under arbitrary joins, we have:

$$F\left(\bigvee S\right) \neq 0.$$

Since F is a proper fuzzy filter, it follows that for any $\mathbb{p} \in S$,

$$F(\mathbb{p}) \neq 0 \quad \text{implies} \quad F(\mathbb{q}) \neq 0 \quad \text{for all} \quad \mathbb{q} \geq \mathbb{p}.$$

This means that for each $\mathbb{p} \in S$, there exists some $\mathbb{q} \in \mathscr{X}$ such that $\mathbb{q} \geq \mathbb{p}$ and $F(\mathbb{q}) \neq 0$.
Thus, since F is closed under arbitrary joins, it contains non-empty sets for every join of elements in $\mathscr{X}$. Therefore, we conclude that:

$$F(\bigvee S) \neq 0 \quad \text{for every non-empty subset } S \subseteq \mathscr{X}.$$

This completes the proof. $\square$

DEFINITION 18. *Let $(\mathscr{X}, \circledast, 0)$ be a BG-algebra. The complement operation $\neg$ on $\mathscr{X}$ is defined for any element $\mathbb{p} \in \mathscr{X}$ such that:*

$$\neg \mathbb{p} = 0 \circledast \mathbb{p},$$

where 0 is the minimum element of the algebra and $\circledast$ is the binary operation of the BG-algebra.

LEMMA 12. *If F is a fuzzy filter on a BG-algebra $(\mathscr{X}, \circledast, 0)$, then F is closed under complements in the sense that if $F(\mathbb{p}) \neq 0$, then $F(\neg \mathbb{p}) \neq 0$ for the complement operation $\neg$ defined as $\neg \mathbb{p} = 0 \circledast \mathbb{p}$.*

PROOF. Let F be a fuzzy filter on the *BG*-algebra $(\mathscr{X}, \circledast, 0)$. Assume $F(\mathbb{p}) \neq 0$ for some $\mathbb{p} \in \mathscr{X}$. This implies $F(\mathbb{p}) > 0$. Using the complement operation defined as $\neg \mathbb{p} = 0 \circledast \mathbb{p}$, we need to show that $F(\neg \mathbb{p}) \neq 0$.
By the Definition 9 of a fuzzy filter, we know:

$$F(0 \circledast \mathbb{p}) \geq F(\mathbb{p}).$$

Since $F(0) = 1$ (non-emptiness) and $F(\mathbb{p}) > 0$, it follows that:

$$F(0 \circledast \mathbb{p}) \geq F(\mathbb{p}) > 0.$$

Now, consider the operation:

$$F(\neg \mathbb{p}) = F(0 \circledast \mathbb{p}) \geq \min\{F(0), F(\mathbb{p})\} = \min\{1, F(\mathbb{p})\} = F(\mathbb{p}) > 0.$$

Thus, we conclude that $F(\neg \mathbb{p}) \neq 0$.
Therefore, the lemma is proven: if $F(\mathbb{p}) \neq 0$, then $F(\neg \mathbb{p}) \neq 0$. $\square$

THEOREM 13. *Let F and G be two fuzzy filters on a BG-algebra $(\mathscr{X}, \circledast, 0)$. If $F \subseteq G$, then F is a fuzzy ideal if and only if G is a fuzzy ideal.*

PROOF. To prove the theorem, we will show both implications:
1. Assume F is a fuzzy ideal. We need to show that G is also a fuzzy ideal.
Since F is a fuzzy ideal, it satisfies the conditions of Definition 14.
Since $F \subseteq G$, we have $G(0) = F(0) = 1$, satisfying the non-emptiness condition for G.
For the closedness under the *BG*-operation:

$$G(\mathbb{p} \circledast \mathbb{q}) \geq F(\mathbb{p} \circledast \mathbb{q}) \geq \min\{F(\mathbb{p}), F(\mathbb{q})\} \geq \min\{G(\mathbb{p}), G(\mathbb{q})\}.$$

For the absorption property, if $G(\mathbb{p} \circledast 0) = 1$, then since $F \subseteq G$, it follows that $F(\mathbb{p} \circledast 0) = 1$, hence $F(\mathbb{p}) = 0$. Therefore, $G(\mathbb{p}) = 0$ as well.
Thus, G satisfies all conditions of a fuzzy ideal.
2. Now, assume G is a fuzzy ideal. We need to show that F is also a fuzzy ideal.
Since G is a fuzzy ideal, it satisfies the same conditions as above:
- **Non-emptiness**: $G(0) = 1$ implies $F(0) = 1$ because $F \subseteq G$.
- **Closedness under the BG-operation**:

$$G(\mathbb{p} \circledast \mathbb{q}) \geq \min\{G(\mathbb{p}), G(\mathbb{q})\} \implies F(\mathbb{p} \circledast \mathbb{q}) \geq \min\{F(\mathbb{p}), F(\mathbb{q})\}.$$

- **Absorption property**: If $G(\mathbb{p} \circledast 0) = 1$, then $F(\mathbb{p}) = 0$.
Thus, F also satisfies all conditions of a fuzzy ideal.
Therefore, we conclude that F is a fuzzy ideal if and only if G is a fuzzy ideal.

□

DEFINITION 19. *Let F be a fuzzy filter on a BG-algebra $(\mathscr{X}, \circledast, 0)$. The fuzzy filter F is said to be a maximal fuzzy filter if there does not exist any proper fuzzy filter G such that $F \subsetneq G$.*

COROLLARY 6. *If F is a proper fuzzy filter on a BG-algebra $\mathscr{X}$, then F can be extended to a maximal fuzzy filter.*

PROOF. Let F be a proper fuzzy filter on a BG-algebra $(\mathscr{X}, \circledast, 0)$. By Definition 15 of a proper fuzzy filter, we know that $F(0) = 1$ and that it satisfies the closure properties of a fuzzy filter. To show that F can be extended to a maximal fuzzy filter, consider the collection of all fuzzy filters on $\mathscr{X}$ that contain F. We denote this collection by $\mathscr{C} = \{G \mid G \text{ is a fuzzy filter on } \mathscr{X} \text{ and } F \subseteq G\}$.
Since the set of fuzzy filters is non-empty (it contains F), we can apply Zorn's Lemma. According to Zorn's Lemma, if every chain (totally ordered subset) in $\mathscr{C}$ has an upper bound in $\mathscr{C}$, then $\mathscr{C}$ contains at least one maximal element.
Let $\{G_\alpha\}_{\alpha \in A}$ be a chain in $\mathscr{C}$. We need to show that there exists an upper bound G in $\mathscr{C}$ such that $F \subseteq G$ and G is also a fuzzy filter.
Define G as follows:

$$G(\mathbb{p}) = \sup_{\alpha \in A} G_\alpha(\mathbb{p}) \quad \text{for all } \mathbb{p} \in \mathscr{X}.$$

We need to check that G satisfies the properties of a fuzzy filter:
1. **Non-emptiness**: Since each G_α is a fuzzy filter, $G_\alpha(0) = 1$ for all α, thus $G(0) = 1$.
2. **Closedness under the BG-operation**: For any $\mathbb{p}, \mathbb{q} \in \mathscr{X}$,

$$G(\mathbb{p} \circledast \mathbb{q}) = \sup_{\alpha \in A} G_\alpha(\mathbb{p} \circledast \mathbb{q}) \geq \min\{G(\mathbb{p}), G(\mathbb{q})\}.$$

3. **Absorption property**: For any $\mathbb{p} \in \mathscr{X}$,

$$G(\mathbb{p} \circledast 0) = G(0) = 1 \implies G(\mathbb{p}) = 0.$$

Thus, G is a fuzzy filter that contains F. Since every chain has an upper bound in $\mathscr{C}$, by Zorn's Lemma, $\mathscr{C}$ contains a maximal element M, which is a maximal fuzzy filter.
Therefore, we conclude that F can be extended to a maximal fuzzy filter on the BG-algebra $\mathscr{X}$. □

EXAMPLE 4.4. *Consider a BG-algebra $\mathscr{X} = \{0, a, b, c\}$ with operations defined as follows:*

$$a \circledast b = c, \quad a \circledast a = 0, \quad b \circledast b = 0, \quad c \circledast c = 0, \quad c \circledast 0 = c.$$

Define the proper fuzzy filter F as:

$$F(a) = 1, \quad F(b) = 1, \quad F(c) = 0.$$

Here, F is a proper fuzzy filter because: - $F(a) \neq 0$ and $F(b) \neq 0$ imply $F(a \circledast b) = F(c) = 0$ satisfies closure under the fuzzy operations. - Additionally, it contains elements such that $F(c) = 0$ indicates that F is not a maximal filter.
Now, to extend F to a maximal fuzzy filter G, we can include all elements of $\mathscr{X}$ such that:

$$G(0) = 0, \quad G(a) = 1, \quad G(b) = 1, \quad G(c) = 1.$$

Thus, G is a maximal fuzzy filter since it includes all non-empty elements of $\mathscr{X}$ and maintains closure under operations. Therefore, F can indeed be extended to the maximal fuzzy filter G.

THEOREM 14. *Let F be a fuzzy filter on a BG-algebra $\mathscr{X}$. If F is closed under finite meets and arbitrary joins, then F is a filter in the classical sense.*

PROOF. We will show that if F is a fuzzy filter on $\mathscr{X}$ that is closed under finite meets and arbitrary joins, then F satisfies the conditions of a filter in the classical sense as per Definition 7.

(1) Non-emptiness:
Since F is a fuzzy filter on $\mathscr{X}$, by definition it contains at least one element. Therefore, $F \neq \emptyset$.
(2) Properness:
By the definition of a fuzzy filter, F does not contain the empty set, ensuring that $\emptyset \notin F$. This satisfies the properness condition.
(3) Upward Inclusion:
Suppose $A \in F$ and $A \subseteq B \subseteq X$.
Since F is closed under arbitrary joins, and since $A \subseteq B$, the element B can be included in F as the join (supremum) of A with itself or with other elements in F. Thus, $B \in F$, satisfying the upward inclusion property.
(4) Finite Intersection Property:
If $A, B \in F$, then by the closure of F under finite meets, $A \cap B \in F$. This satisfies the finite intersection property.

Since F meets all four conditions of Definition 7, it follows that F is a filter in the classical sense. □

THEOREM 15. *Let F be a fuzzy filter on a BG-algebra $\mathscr{X}$. If F is closed under finite meets, then F is a fuzzy filter in the sense that for any $\mathbb{p}, \mathbb{q} \in F$, we have $F(\mathbb{p} \circledast \mathbb{q}) \neq 0$.*

PROOF. To prove that F is a fuzzy filter, we need to show that if $\mathbb{p}$ and $\mathbb{q}$ are in F (i.e., $F(\mathbb{p}) \neq 0$ and $F(\mathbb{q}) \neq 0$), then $F(\mathbb{p} \circledast \mathbb{q}) \neq 0$. Since F is closed under finite meets, we can consider the meet $F(\mathbb{p} \circledast \mathbb{q})$. By the definition of closure under finite meets, we have:

$$F(\mathbb{p} \circledast \mathbb{q}) \geq \min\{F(\mathbb{p}), F(\mathbb{q})\}.$$

Because $F(\mathbb{p}) \neq 0$ and $F(\mathbb{q}) \neq 0$, it follows that:

$$\min\{F(\mathbb{p}), F(\mathbb{q})\} > 0.$$

Hence, we can conclude that:

$$F(\mathbb{p} \circledast \mathbb{q}) \neq 0.$$

Therefore, F is a fuzzy filter as it satisfies the condition that for any $\mathbb{p}, \mathbb{q} \in F$, $F(\mathbb{p} \circledast \mathbb{q}) \neq 0$. □

The further study investigates some more properties and results of fuzzy filters on BG-algebras. It comprises corollary, theorems and examples showing the action and relations on fuzzy filters. Interestingly, we prove that a fuzzy filter, which is closed under arbitrary joins and finite meets, is maximal; that is, it can not be enlarged to a fuzzy filter without losing its filter qualities. An additional significant result is that, if two fuzzy filters are maximal, they must actually be the same object. In addition, this work deals with closure properties of fuzzy filters, showing that a proper fuzzy filter that is closed under finitely meets is also closed under finitely joins, and if a proper fuzzy filter is closed under operation $\circledast$, it acts as a classical filter. The examples and proofs presented shows these properties and how the maximal fuzzy filters behave in *BG*-algebras.

COROLLARY 7. *If F is a fuzzy filter on a BG-algebra $\mathscr{X}$ that is closed under arbitrary joins and finite meets, then F is a maximal fuzzy filter.*

PROOF. To show that F is a maximal fuzzy filter, we need to demonstrate that there are no proper extensions of F that also satisfy the properties of a fuzzy filter on $\mathscr{X}$.
Assume G is a proper extension of F such that $F \subsetneq G$. Since G is a fuzzy filter, it must also be closed under finite meets and arbitrary joins.
1. **Closure under Finite Meets**: Since F is closed under finite meets, for any $\mathbb{p}, \mathbb{q} \in F$, we have $F(\mathbb{p} \circledast \mathbb{q}) \neq 0$. Because G contains F, it also holds that $G(\mathbb{p} \circledast \mathbb{q}) \neq 0$ for any $\mathbb{p}, \mathbb{q} \in G$.
2. **Closure under Arbitrary Joins**: By the property of arbitrary joins, for any subset $S \subseteq G$, there exists an element $\bigvee S \in G$ such that $\mathbb{p} \leq \bigvee S$ for all $\mathbb{p} \in S$. Since F is closed under arbitrary joins, we have:

$$F(\bigvee S) \geq \min\{F(\mathbb{p}) : \mathbb{p} \in S\} > 0,$$

implying that G is also closed under arbitrary joins.
Since G extends F while satisfying the properties of a fuzzy filter, G must include all non-empty fuzzy subsets of $\mathscr{X}$ due to the closure properties.
However, this would contradict the assumption that F is a proper subset of G unless $G = F$. Hence, F cannot be properly extended without losing its properties as a fuzzy filter.
Thus, we conclude that F is a maximal fuzzy filter. $\square$

LEMMA 16. *Let F and G be two fuzzy filters on a BG-algebra $\mathscr{X}$. If F and G are both maximal, then $F = G$.*

PROOF. Assume for contradiction that F and G are distinct fuzzy filters, meaning there exists an element $\mathbb{p} \in G$ such that $\mathbb{p} \notin F$.
Since F is a maximal fuzzy filter, it cannot be properly extended while preserving the properties of a fuzzy filter. Therefore, the set $F \cup \{\mathbb{p}\}$ would also be a fuzzy filter on $\mathscr{X}$.
1. **Closure Under Operations**: - Since F is a fuzzy filter, it is closed under finite meets and arbitrary joins. Thus, for any $\mathbb{q} \in F$, we have $F(\mathbb{q}) \neq 0$ and hence $F(\mathbb{p} \circledast \mathbb{q}) \neq 0$ must hold for all $\mathbb{q} \in F$. - Similarly, since G is also a fuzzy filter, it is closed under finite meets and arbitrary joins. Therefore, we have $G(\mathbb{p}) \neq 0$ for all $\mathbb{p} \in G$.
2. **Contradiction**: - Since F is closed under operations and includes elements of G, adding $\mathbb{p}$ to F allows us to create a fuzzy filter that includes all elements of G, thus contradicting the maximality of F. Therefore, F must contain all elements of G.
By a symmetric argument, if G is distinct from F, then G must also contain all elements of F. Hence, we conclude that:

$$F = G.$$

Thus, if both F and G are maximal fuzzy filters on $\mathscr{X}$, it follows that $F = G$. $\square$

EXAMPLE 4.5. *Let $\mathscr{X} = \{0, a, b, c\}$ be a BG-algebra with operations defined as follows:*

$$a \circledast b = c, \quad a \circledast a = 0, \quad b \circledast b = 0, \quad c \circledast c = 0, \quad c \circledast 0 = c.$$

Define two maximal fuzzy filters F and G on $\mathscr{X}$ as follows:

$$F(0) = 0, \quad F(a) = 1, \quad F(b) = 1, \quad F(c) = 1,$$

$$G(0) = 0, \quad G(a) = 1, \quad G(b) = 1, \quad G(c) = 1.$$

Here, both F and G are maximal fuzzy filters because they include all non-empty elements of $\mathscr{X}$ and satisfy the closure properties of fuzzy filters.
However, since both filters assign the same values to all elements of $\mathscr{X}$, we have $F = G$. This demonstrates that if two fuzzy filters on a BG-algebra are both maximal, then they must be equal.

THEOREM 17. *Let F be a proper fuzzy filter on a BG-algebra $\mathscr{X}$. If F is closed under the operation $\circledast$, then F is a filter in the classical sense.*

PROOF. To prove that F is a filter in the classical sense, we need to show that for any $\mathbb{p}, \mathbb{q} \in F$, the element $F(\mathbb{p} \circledast \mathbb{q}) \neq 0$.
Since F is a proper fuzzy filter, we have $F(\mathbb{p}) \neq 0$ and $F(\mathbb{q}) \neq 0$.
By the Definition 15 of a proper fuzzy filter, it satisfies the closure under the operation $\circledast$.
Thus, we can write:

$$F(\mathbb{p}) \neq 0 \quad \text{and} \quad F(\mathbb{q}) \neq 0.$$

By the closure property of F, we have:

$$F(\mathbb{p} \circledast \mathbb{q}) \neq 0.$$

This shows that F satisfies the condition for being a filter in the classical sense, as it contains all finite meets of its elements.
Hence, F is a filter in the classical sense. $\square$

THEOREM 18. *Let F be a proper fuzzy filter on a BG-algebra $\mathscr{X}$. If F is closed under finite meets, then F is also closed under finite joins.*

PROOF. Let F be a proper fuzzy filter on a *BG*-algebra $\mathscr{X}$ that is closed under finite meets. We need to show that F is also closed under finite joins.
Let $\mathbb{p}_1, \mathbb{p}_2, \ldots, \mathbb{p}_n \in F$. By the Definition 15 of a proper fuzzy filter, we have $F(\mathbb{p}_i) \neq 0$ for each $i = 1, 2, \ldots, n$.
Since F is closed under finite meets, we can consider the finite meet of these elements:

$$\mathbb{p}_{\text{meet}} = \mathbb{p}_1 \circledast \mathbb{p}_2 \circledast \cdots \circledast \mathbb{p}_n.$$

By the closure property of F, we have:

$$F(\mathbb{p}_{\text{meet}}) \neq 0.$$

Now, since the join $\bigvee\{\mathbb{p}_1, \mathbb{p}_2, \ldots, \mathbb{p}_n\}$ is defined as the least upper bound of the set, we can assert:

$$\bigvee\{\mathbb{p}_1, \mathbb{p}_2, \ldots, \mathbb{p}_n\} \geq \mathbb{p}_i \quad \text{for all } i.$$

Therefore, by the properties of fuzzy filters, we conclude that:

$$F\left(\bigvee\{\mathbb{p}_1, \mathbb{p}_2, \ldots, \mathbb{p}_n\}\right) \neq 0.$$

Hence, F is closed under finite joins. This completes the proof. $\square$

THEOREM 19. *If F is a proper fuzzy filter on a BG-algebra $\mathscr{X}$ such that $F(\mathbb{p}) \geq F(\mathbb{q})$ for all $\mathbb{p} \geq \mathbb{q}$, then F is a fuzzy ideal.*

PROOF. To show that F is a fuzzy ideal, we need to demonstrate that it satisfies the following conditions:
1. For any $\mathbb{p}, \mathbb{q} \in \mathscr{X}$, if $F(\mathbb{p}) \neq 0$, then $F(\mathbb{p} \circledast \mathbb{q}) \neq 0$.
2. If $F(\mathbb{p}) \neq 0$ for some $\mathbb{p} \in \mathscr{X}$, then $F(\mathbb{q}) \neq 0$ for all $\mathbb{q} \leq \mathbb{p}$.
Step 1: Closure under the operation $\circledast$
Since F is a proper fuzzy filter, we have $F(\mathbb{p}) \neq 0$ for some $\mathbb{p} \in \mathscr{X}$. For any $\mathbb{q} \in \mathscr{X}$ where $\mathbb{q} \leq \mathbb{p}$, the condition $F(\mathbb{p}) \geq F(\mathbb{q})$ implies that:

$$F(\mathbb{p} \circledast \mathbb{q}) \geq F(\mathbb{p}) \circledast F(\mathbb{q}) \neq 0.$$

Hence, $F(\mathbb{p} \circledast \mathbb{q}) \neq 0$.
Step 2: Closure under lower bounds
For any $\mathbb{p} \in \mathscr{X}$, if $F(\mathbb{p}) \neq 0$, then for any $\mathbb{q} \leq \mathbb{p}$, we have:

$$F(\mathbb{q}) \geq F(\mathbb{p}) > 0.$$

Therefore, $F(\mathbb{q}) \neq 0$.
Since both conditions for being a fuzzy ideal are satisfied, we conclude that F is a fuzzy ideal. □

EXAMPLE 4.6. *Consider the BG-algebra $\mathscr{X} = \{0, a, b, c\}$ with operations defined as follows:*

$$a \circledast b = c, \quad a \circledast a = 0, \quad b \circledast b = 0, \quad c \circledast 0 = c.$$

Define the fuzzy filter F as:

$$F(a) = 1, \quad F(b) = 1, \quad F(c) = 0.$$

Now, we verify the condition $F(\mathbb{p}) \geq F(\mathbb{q})$ for all $\mathbb{p} \geq \mathbb{q}$:
- Since $F(a) = 1$ and $F(b) = 1$, we have $F(a) \geq F(b)$. - $F(c) = 0$ and since $c \leq a$ and $c \leq b$, it follows that $F(a) \geq F(c)$ and $F(b) \geq F(c)$.
To show that F is a fuzzy ideal, we check the conditions:
*1. **Closure under the operation $\circledast$**: For any $\mathbb{p}, \mathbb{q} \in \mathscr{X}$ such that $F(\mathbb{p}) \neq 0$ (let's take $\mathbb{p} = a$ and $\mathbb{q} = b$):*

$$F(a \circledast b) = F(c) = 0.$$

However, for a and c, since $F(a) = 1$ and $F(c) = 0$, we still have $F(a \circledast c) = F(b) = 1$.
*2. **Closure under lower bounds**: For any $\mathbb{p} \in \mathscr{X}$ with $F(\mathbb{p}) \neq 0$ (for instance, $\mathbb{p} = a$), if $\mathbb{q} \leq \mathbb{p}$ (like c):*

$$F(c) = 0 \quad \text{and} \quad F(a) \geq F(c).$$

Since both conditions of a fuzzy ideal are satisfied, we conclude that F is indeed a fuzzy ideal.

DEFINITION 20. *Let $(\mathscr{X}, \circledast, 0)$ be a BG-algebra and let $\mathbb{p} \in \mathscr{X}$. The principal fuzzy filter generated by $\mathbb{p}$, denoted $F_{\mathbb{p}}$, is defined as:*

$$F_{\mathbb{p}} = \{\mathbb{q} \in \mathscr{X} : F(\mathbb{q}) \geq F(\mathbb{p})\}.$$

This set contains all elements $\mathbb{q}$ in $\mathscr{X}$ for which the fuzzy value of $\mathbb{q}$ is greater than or equal to the fuzzy value of $\mathbb{p}$ under the filter F.

THEOREM 20. *If F is a maximal fuzzy filter on a BG-algebra $\mathscr{X}$, then F is uniquely determined by its principal filters.*

PROOF. Let F be a maximal fuzzy filter on a *BG*-algebra $\mathscr{X}$. To prove that F is uniquely determined by its principal filters, assume there are two maximal fuzzy filters F_1 and F_2 such that both contain the same principal filter generated by some $\mathbb{p} \in \mathscr{X}$. We want to show that $F_1 = F_2$.
Since F_1 and F_2 are maximal fuzzy filters, for any element $\mathbb{q} \in \mathscr{X}$:
- If $F_1(\mathbb{q}) \neq 0$, then $\mathbb{q}$ is included in F_1.
- If $F_2(\mathbb{q}) \neq 0$, then $\mathbb{q}$ is included in F_2.
Now, consider an element $\mathbb{r} \in \mathscr{X}$ such that $F_1(\mathbb{r}) \neq 0$. Since F_1 is maximal, either $F_2(\mathbb{r}) \neq 0$ or there exists a $\mathbb{q}$ such that $F_2(\mathbb{q}) = 0$. However, because both filters contain the same principal filters, F_1 must also include all elements that F_2 does, and vice versa. Therefore, for every $\mathbb{r}$ in F_1, $F_2(\mathbb{r}) \neq 0$ must hold, leading us to conclude that F_1 and F_2 cannot differ.
Thus, we have shown that if F_1 and F_2 have the same principal filters, then $F_1 = F_2$. Therefore, F is uniquely determined by its principal filters. □

COROLLARY 8. *Every fuzzy filter on a BG-algebra $\mathscr{X}$ can be generated by its principal filters.*

PROOF. Let F be a fuzzy filter on a *BG*-algebra $\mathscr{X}$. By Definition 20, the filter F consists of all elements in $\mathscr{X}$ that satisfy certain conditions. Since every element in F can be expressed in terms of its fuzzy values relative to other elements, we can assert that:

$$F = \bigcup_{\mathbb{p} \in F} F_{\mathbb{p}}.$$

This union indicates that every element of F can be represented as an element of some principal filter $F_{\mathbb{p}}$, which in turn means that F is generated by its principal filters.
Thus, we conclude that every fuzzy filter on a *BG*-algebra can be generated by its principal filters. □

LEMMA 21. *Let F be a fuzzy filter on a BG-algebra $\mathscr{X}$. If F is a proper filter, then for any $\mathbb{p} \in F$, the set $\{\mathbb{p} \circledast \mathbb{q} \mid \mathbb{q} \in F\}$ is also contained in F.*

PROOF. Let $\mathbb{p} \in F$ be an arbitrary element of the fuzzy filter F. By the Definition 9 of a fuzzy filter, F is closed under the operation $\circledast$. This means that if $F(\mathbb{p}) \neq 0$ and $F(\mathbb{q}) \neq 0$ for any $\mathbb{q} \in F$, then $F(\mathbb{p} \circledast \mathbb{q}) \neq 0$.
Since F is a proper fuzzy filter, we have $F(\mathbb{p}) \neq 0$. Furthermore, for any $\mathbb{q} \in F$, it follows that $F(\mathbb{q}) \neq 0$ as well. Therefore, applying the closure property of the filter under the operation $\circledast$, we obtain:

$$F(\mathbb{p} \circledast \mathbb{q}) \neq 0 \quad \text{for all } \mathbb{q} \in F.$$

Thus, the element $\mathbb{p} \circledast \mathbb{q}$ is also contained in F for every $\mathbb{q} \in F$. Therefore, we can conclude that:

$$\{\mathbb{p} \circledast \mathbb{q} \mid \mathbb{q} \in F\} \subseteq F.$$

Consequently, this shows that the set formed by the operation of $\mathbb{p}$ with all elements of F is contained in F, completing the proof. □

THEOREM 22. *Let F be a fuzzy filter on a BG-algebra $\mathscr{X}$ such that for every $\mathbb{p} \in F$, there exists a $\mathbb{q} \in F$ such that $\mathbb{p} \circledast \mathbb{q} = 0$. Then F is not a proper filter.*

PROOF. Suppose, for the sake of contradiction, that F is a proper filter. By Definition 15 of a proper filter, this means that there exists at least one element $\mathbb{r} \in F$ such that $F(\mathbb{r}) \neq 0$.
According to the hypothesis of the theorem, for every $\mathbb{p} \in F$, there exists some $\mathbb{q} \in F$ such that:

$$\mathbb{p} \circledast \mathbb{q} = 0.$$

Now, taking $\mathbb{p} = \mathbb{r}$, we can find a $\mathbb{q} \in F$ such that:

$$\mathbb{r} \circledast \mathbb{q} = 0.$$

Since $\mathbb{r}$ is an element of the proper filter F and F is closed under the operation $\circledast$, we would expect $F(\mathbb{r} \circledast \mathbb{q}) \neq 0$ for any non-zero $\mathbb{r}$ and $\mathbb{q}$. However, we have established that:

$$F(\mathbb{r} \circledast \mathbb{q}) = F(0) = 0.$$

This creates a contradiction, as a proper filter must contain non-zero elements resulting from the operation on other elements in the filter. Therefore, our assumption that F is a proper filter must be incorrect.
Hence, we conclude that F is not a proper filter. □

EXAMPLE 4.7. *Consider a BG-algebra $\mathscr{X} = \{0, a, b, c\}$ with operations defined as follows:*

$$a \circledast a = 0, \quad b \circledast b = 0, \quad c \circledast 0 = c, \quad a \circledast b = c.$$

Define the fuzzy filter F as:

$$F(a) = 1, \quad F(b) = 1, \quad F(c) = 0.$$

In this case, we observe that:

$$F(a \circledast a) = F(0) = 0,$$

which means $a \in F$ has a corresponding $q = a$ such that:

$$a \circledast a = 0.$$

Similarly, for b:

$$F(b \circledast b) = F(0) = 0,$$

indicating that $b \in F$ also satisfies the condition with $q = b$.
Thus, since both a and b yield zero when combined with themselves, we conclude that F is not a proper filter in this BG-algebra. This example illustrates the theorem as all elements in F can be combined with themselves to produce the zero element, thus contradicting the requirements for a proper filter.

5. FUZZY PRIME, CLOSED, AND SYMMETRIC FILTERS IN *BG*-ALGEBRAS

This section explains the definition and some properties of fuzzy prime filters in the context of *BG*-algebras. A fuzzy filter F on a *BG*-algebra X is called a fuzzy prime filter on X, if $F(a) > 0, F(b) > 0 \Rightarrow F(a \circledast b) = 0$. A lemma and its proof were given to prove that if F satisfies this above condition, then it is a fuzzy prime filter. We further considered its examples that embody the definition, such that based on algebraic operations of the fuzzy filter we can ensure that it is a fuzzy prime filter as well. In addition, we defined closed fuzzy filters, bounded fuzzy filters, and symmetric fuzzy filters, and provided some theorems establishing the conditions under which these filters respectively preserve some of the closure properties and symmetry. Moreover, we studied the complement of a fuzzy filter, proved that the complement of a proper fuzzy filter is also a fuzzy filter. This provides the basis for working through the properties of fuzzy filters in *BG*-algebras, their closure, boundedness, propensity towards symmetry etc, providing import materials for further theoretical studies in fuzzy algebraic structures.

DEFINITION 21. *Let $(\mathscr{X}, 0, \circledast)$ be a BG-algebra. A fuzzy filter $F : \mathscr{X} \to [0,1]$ is called a fuzzy prime filter if it satisfies the following condition:*

For any $a, b \in \mathscr{X}$, if $F(a) > 0$ and $F(b) > 0$, then $F(a \circledast b) = 0$.

LEMMA 23. *Let $(\mathscr{X}, 0, \circledast)$ be a BG-algebra and F a fuzzy filter. If $F(a) > 0$ and $F(b) > 0$ imply $F(a \circledast b) = 0$, then F is a fuzzy prime filter.*

PROOF. To show that F is a fuzzy prime filter, we need to verify the following condition: for any $\mathbb{p}, \mathbb{q} \in \mathscr{X}$, if $F(\mathbb{p}) > 0$ and $F(\mathbb{q}) > 0$, then $F(\mathbb{p} \circledast \mathbb{q}) = 0$.
Given that $F(a) > 0$ and $F(b) > 0$ implies $F(a \circledast b) = 0$, we start by assuming $F(\mathbb{p}) > 0$ and $F(\mathbb{q}) > 0$.
1. Assume $F(\mathbb{p}) > 0$.
2. Assume $F(\mathbb{q}) > 0$.
By the given hypothesis, we can conclude:

$$F(\mathbb{p} \circledast \mathbb{q}) = 0.$$

This establishes that F does not take on positive values for the operation $\circledast$ applied to any pair of elements in F, thereby confirming that F is a fuzzy prime filter.
Thus, we have shown that if $F(a) > 0$ and $F(b) > 0$ lead to $F(a \circledast b) = 0$, then F is indeed a fuzzy prime filter. □

EXAMPLE 5.1. *Consider the BG-algebra $\mathscr{X} = \{0, a, b, c\}$ with operations defined as follows:*

$$a \circledast b = c, \quad a \circledast a = 0, \quad b \circledast b = 0, \quad c \circledast c = 0, \quad c \circledast 0 = c.$$

Define the fuzzy filter F on $\mathscr{X}$ as:

$$F(a) = 1, \quad F(b) = 1, \quad F(c) = 0.$$

Now, let's verify that F is a fuzzy prime filter:
- We have $F(a) > 0$ and $F(b) > 0$.
- According to the definition, we must check $F(a \circledast b)$:

$$F(a \circledast b) = F(c) = 0.$$

Since $F(a) > 0$ and $F(b) > 0$ imply $F(a \circledast b) = 0$, we can conclude that F is indeed a fuzzy prime filter.
Thus, the fuzzy filter F defined above satisfies the condition of being a fuzzy prime filter in the BG-algebra $\mathscr{X}$.

EXAMPLE 5.2. *Consider the BG-algebra $\mathscr{X} = \{0, 1\}$ with operations defined as follows:*

$$0 \circledast 0 = 0, \quad 0 \circledast 1 = 0, \quad 1 \circledast 0 = 0, \quad 1 \circledast 1 = 0.$$

Let F be a fuzzy filter where:

$$F(1) = 1 \quad \text{and} \quad F(0) = 0.$$

This filter is closed under finite meets and unions, and it satisfies the conditions for being an upper bounded filter.

REMARK 1. *The results presented herein illustrate the interplay between BG-algebras and fuzzy filters. The concepts of closure, boundedness, and symmetry provide a robust framework for further exploration of fuzzy algebraic structures.*

DEFINITION 22. *A fuzzy filter F on a BG-algebra $\mathscr{X}$ is said to be closed under a binary operation $\circledast$ if for any elements $\mathbb{p}, \mathbb{q} \in \mathscr{X}$, whenever $F(\mathbb{p}) \neq 0$ and $F(\mathbb{q}) \neq 0$, it follows that $F(\mathbb{p} \circledast \mathbb{q}) \neq 0$.*

DEFINITION 23. *A fuzzy filter F on a BG-algebra $\mathscr{X}$ is called bounded if there exists a minimum element $0 \in \mathscr{X}$ such that $F(0) = 1$, meaning that the fuzzy filter contains the minimum element of the algebra.*

DEFINITION 24. *A fuzzy filter F on a BG-algebra $\mathscr{X}$ is said to be symmetric if for any $\mathbb{p} \in \mathscr{X}$, $F(\mathbb{p}) = F(\neg\mathbb{p})$, where $\neg\mathbb{p}$ denotes the complement of $\mathbb{p}$ in the BG-algebra.*

THEOREM 24. *Let F be a fuzzy filter on a BG-algebra $\mathscr{X}$ that is closed under the operation $\circledast$. Then for any $\mathbb{p}, \mathbb{q} \in \mathscr{X}$ with $F(\mathbb{p}) \neq 0$ and $F(\mathbb{q}) \neq 0$, we have $F(\mathbb{p} \circledast \mathbb{q}) \neq 0$.*

PROOF. Assume that F is a fuzzy filter on $\mathscr{X}$ and is closed under $\circledast$. Let $\mathbb{p}, \mathbb{q} \in \mathscr{X}$ such that $F(\mathbb{p}) \neq 0$ and $F(\mathbb{q}) \neq 0$. By the closure property, $\mathbb{p} \circledast \mathbb{q} \in \mathscr{X}$ and $F(\mathbb{p} \circledast \mathbb{q}) \neq 0$. Hence, the theorem holds. □

THEOREM 25. *If F is a fuzzy filter on a BG-algebra $\mathscr{X}$ and F is bounded, then $0 \in F$ and $F(0) = 1$.*

PROOF. By the Definition 23 of boundedness, there exists a minimum element $0 \in \mathscr{X}$ such that $F(0) = 1$. This implies that the fuzzy filter F includes the minimum element of the *BG*-algebra, ensuring that $0 \in F$ and is fully included with maximum membership value. □

THEOREM 26. *Let F be a symmetric fuzzy filter on a BG-algebra $\mathscr{X}$. Then for any $\mathbb{p} \in \mathscr{X}$, $F(\mathbb{p}) = F(\neg\mathbb{p})$.*

PROOF. Assume that F is a symmetric fuzzy filter, so by Definition 24, for any $\mathbb{p} \in \mathscr{X}$, $F(\mathbb{p}) = F(\neg\mathbb{p})$. This implies that the membership value assigned to an element and its complement in the *BG*-algebra is identical. Therefore, the theorem is true by the definition of symmetry. □

PROPOSITION 4. *If F is a proper fuzzy filter on a BG-algebra $\mathscr{X}$, then F contains at least one element $x \in \mathscr{X}$ such that $x \circledast 0 = x$.*

PROOF. Let F be a proper fuzzy filter on a *BG*-algebra $\mathscr{X}$. By the Definition 15 of a proper fuzzy filter, F is non-empty and does not contain the zero element 0 as its sole member.
To prove the proposition, we need to show that there exists an element $x \in \mathscr{X}$ such that $x \circledast 0 = x$ and $F(x) \neq 0$.
Since F is proper, there must be at least one element $x \in \mathscr{X}$ for which $F(x) \neq 0$. Consider any such element $x \in F$. In a *BG*-algebra, the operation $\circledast$ satisfies the property that $x \circledast 0 = x$. Hence, this element x exists in F, and x satisfies $x \circledast 0 = x$.
Therefore, F contains at least one element $x \in \mathscr{X}$ such that $x \circledast 0 = x$, which completes the proof. □

COROLLARY 9. *Every fuzzy filter in a BG-algebra is non-empty.*

PROOF. Let F be a fuzzy filter on a *BG*-algebra $\mathscr{X}$. By the Definition 9 of a fuzzy filter, F is a mapping from $\mathscr{X}$ to the interval $[0,1]$, where $F(x)$ represents the degree of membership of the element x in the filter F.
Assume for contradiction that F is empty, meaning that there are no elements $x \in \mathscr{X}$ for which $F(x)$ is greater than zero. In other words, if F were empty, it would mean $F(x) = 0$ for all $x \in \mathscr{X}$.
However, for F to be a proper fuzzy filter, it must satisfy certain properties, including the condition that there exists at least one element $x \in \mathscr{X}$ with $F(x) \neq 0$ (i.e., F should not be empty). This is because a fuzzy filter must be able to represent a collection of elements in the *BG*-algebra with non-zero membership values.
Thus, the assumption that F is empty leads to a contradiction, implying that every fuzzy filter in a *BG*-algebra must be non-empty.
Therefore, the corollary is proved. □

LEMMA 27. *For every fuzzy filter F on a BG-algebra $\mathscr{X}$, the complement $F^c = \{x \in \mathscr{X} : F(x) = 0\}$ is also a fuzzy filter if and only if F is a proper filter.*

PROOF. Let F be a fuzzy filter on a *BG*-algebra $\mathscr{X}$. The complement F^c is defined as the set of elements in $\mathscr{X}$ where the membership value in F is zero, i.e.,

$$F^c = \{x \in \mathscr{X} : F(x) = 0\}.$$

We need to prove that F^c is also a fuzzy filter if and only if F is a proper filter.
(1) Necessity: Assume that F is a proper fuzzy filter. A fuzzy filter F is considered proper if it does not contain all elements of $\mathscr{X}$ with membership value 1 (i.e., $F(x) < 1$ for at least one $x \in \mathscr{X}$).
If F is a proper filter, then there exist elements in $\mathscr{X}$ for which $F(x) = 0$. This means the complement F^c is non-empty. To check whether F^c is a fuzzy filter, it must satisfy the conditions for being a fuzzy filter:
1. **Non-emptiness**: Since F is proper, F^c is non-empty.
2. **Closure under the operation $\circledast$**: If $x, y \in F^c$, then $F(x) = 0$ and $F(y) = 0$. For F^c to be a filter, we need $F(x \circledast y) = 0$. This is true because if $F(x) = 0$ and $F(y) = 0$, then $F(x \circledast y)$ should also be 0 (closure under the operation is preserved in fuzzy filters).
Thus, F^c satisfies the conditions of a fuzzy filter.
(2) Sufficiency: Conversely, assume that the complement F^c is a fuzzy filter. This implies that there are elements in $\mathscr{X}$ where $F(x) = 0$, meaning that F cannot have all elements with membership value 1 (otherwise, F^c would be empty).
Hence, if F^c is a fuzzy filter, then F must be a proper filter.
Therefore, F^c is a fuzzy filter if and only if F is a proper filter. □

EXAMPLE 5.3. *Let $\mathscr{X} = \{0, a, b, c\}$ be a BG-algebra with the operation $\circledast$ defined as follows:*

$\circledast$	0	a	b	c
0	0	0	0	0
a	0	a	c	b
b	0	c	b	a
c	0	b	a	c

Define a fuzzy filter F on $\mathscr{X}$ with the membership function F given by:

$$F(0) = 0, \quad F(a) = 0.5, \quad F(b) = 1, \quad F(c) = 0.$$

Complement of the Fuzzy Filter F: The complement F^c is defined as the set of elements in $\mathscr{X}$ where $F(x) = 0$. Thus, the complement set is:

$$F^c = \{x \in \mathscr{X} : F(x) = 0\} = \{0, c\}.$$

Verification: We need to check if F^c forms a fuzzy filter:
*1. **Non-emptiness**: $F^c = \{0, c\}$ is non-empty.*
*2. **Closure under the operation $\circledast$**: - $0 \circledast 0 = 0 \in F^c$. - $0 \circledast c = 0 \in F^c$. - $c \circledast 0 = 0 \in F^c$. - $c \circledast c = c \in F^c$.*
Since the complement $F^c = \{0, c\}$ is closed under the operation $\circledast$, it is a fuzzy filter.
Proper Filter: The original fuzzy filter F is proper because it does not assign a membership value of 1 to every element in $\mathscr{X}$. Thus, the complement F^c satisfies the conditions for being a fuzzy filter.

6. CONCLUSION

The study of *BG*-algebra and its generalization to include concepts from fuzzy logic give complete algebraic systems where both classical and fuzzy members can be treated. Building on fuzzy filters, ideals, and subalgebras will produce a systematic that can model a wide variety of algebraic systems in a flexible way. Introducing fuzziness into *BG*-algebras complex extensions, while preserving fundamental rules of operation, gives the integration of classical algebra and fuzzy logic. The polarity principle is applied here. This paper shows how *BG*-algebras can work with both crisper and fuzzier creations plausibly, laying the basis for more theoretical developments and applications within algebraic frameworks.
Looking at filters–especially fuzzy filters–from *BG*-algebras point of view gives one a new insight and many interesting objects.

Classical confronts allow us to understand substructures better, but by transmuting to fuzzy conciliators complication and adaptability added stridently on to it. Where membership levels are concerned, as well as closure under operations, both intersection and unification conditions are fulfilled–fuzzy filters display an inherent vigor and practicality. Furthermore, the process of finding maximal fuzzy filters their movement under Zorn's lemma shows clearly the analysis's depth in concept, it opens up new possibilities for exacting fuzzy algebraic systems.

Finally, by analyzing exotic objects such as fuzzy prime filters and their interaction with *BG*-operations, the logical and algebraic coherence of the system is strengthened. The symmetric, boundary and complementary natures of fuzzy filters plus their connection with ideals offer a rich but delicate balance between classical rigidity and fuzzy lightness. These constructs not only enrich the theory of *BG*-algebra but also offer a way to apply tools from computer-assisted decision making, fuzzy logic application and computational algebra into practice.

In conclusion, the wide-ranging algebraic systems created by combining *BG*-algebra with fuzzy set theory are rich soil for further mathematical investigation and application. The various structural characteristics, the results of our theoretical investigations, and concrete examples are all helpful in understanding these algebras more deeply. Future research should be able to build on this foundation by relating *BG*-algebras to other algebraic structures, developing practical computer implementations, and using the results obtained to solve real-world problems at the interface between fuzziness and algebraic structure. The interaction between classical algebra and fuzziness under the *BG*-algebraic framework looks promising for both further theoretical development and actual uses.

6.1 Future Work

Building upon this study's findings and insights, several promising future research paths in *BG*-algebras and fuzzy extensions were identified. Exploration of connections with other algebraic systems could investigate relationships between *BG*-algebras and structures like lattice theory, residuated lattices or generalizations. Computationally modeling *BG*-algebras and fuzzy extensions would enable practical applications in decision making, AI and data analysis by facilitating uncertainty handling. Investigating fuzzy filters and ideals in *BG*-algebras for decision processes, information retrieval and domains where fuzzy logic plays a crucial role could broaden practical application. Integrating *BG*-algebras with multi-valued or intuitionistic fuzzy logic could enhance expressiveness in complex logical scenarios. Examining behavior under dynamic or temporal conditions like evolving membership degrees or time-dependent operations could provide novel insights and applications in temporal logic and systems theory. Advanced study of fuzzy prime and maximal filters, particularly related to their extremal behavior and structural significance, could yield valuable contributions. Extending study into topological and geometric frameworks may open interdisciplinary opportunities through fuzzy topological spaces or geometric interpretations. Empirical studies or case validations in domains like computational intelligence, pattern recognition or automated reasoning could substantiate theoretical relevance. Expanding the framework to accommodate alternative or generalized membership functions could offer greater flexibility applicability in complex systems modeling with varying degrees of fuzziness. These avenues highlight potential for theoretical advancement and practical innovation underscoring versatility and importance in modern algebraic research and applications.

7. CONFLICTS OF INTEREST

The authors declare that there is no conflict of interest regarding the publication of this paper.

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9. REFERENCES

[1] B. Ahmad. Fuzzy bci-algebras. *Journal of Fuzzy Mathematics*, 1(2):445–452, 1993.

[2] S. S. Ahn and H. D. Lee. Fuzzy subalgebras of bg-algebras. *Communications of the Korean Mathematical Society*, 19(2):243–251, 2004.

[3] M. Akram. *m*-polar fuzzy sets and their applications in bck/bci-algebras. *Journal of Intelligent and Fuzzy Systems*, 36(5):5303–5313, 2019.

[4] A. Al-Masarwah. Bipolar fuzzy subalgebras and their extensions. *Fuzzy Information and Engineering*, 12(4):398–411, 2020.

[5] S. R. Barbhuiya. (α,β)-doubt fuzzy ideals of bg-algebras. *International Journal of Advances in Mathematics*, 2018(1):83–94, 2018.

[6] D. K. Basnet and L. B. Singh. On $(\in,\in\vee q)$-fuzzy ideals in bg-algebras. *Journal of Fuzzy Mathematics*, 30(2):95–108, 2022.

[7] G. Büyüközkan and G. Çifçi. A novel hybrid mcdm approach based on fuzzy dematel, fuzzy anp, and fuzzy topsis to evaluate green suppliers. *Expert Systems with Applications*, 39(3):3000–3011, 2012.

[8] J. Castelló-Sirvent, L. Higuera, J. J. García, J. L. G. Sánchez, and A. D. Abreu. Bibliometric analysis of fuzzy analytic hierarchy process (fahp) and its applications in decision making. *Mathematics*, 10(6):1748, 2022.

[9] B. A. Davey and H. A. Priestley. *Introduction to Lattices and Order*. Cambridge University Press, 2nd edition, 2002.

[10] J. A. Goguen. *L*-fuzzy sets. *Journal of Mathematical Analysis and Applications*, 18(1):145–174, 1967.

[11] S. M. Hong and Y. B. Jun. Anti fuzzy ideals in bck-algebras. *Kyungpook Math. J.*, 38:145–150, 1998.

[12] Q. P. Hu and X. Li. On bch-algebras. *Math. Seminar Notes*, 11:313–320, 1983.

[13] Q. P. Hu and X. Li. On proper bch-algebras. *Math. Japonica*, 30:659–661, 1985.

[14] F. Y. Huang. Another definition of fuzzy bci-algebras – in chinese. In *Selected Papers on BCK and BCI-Algebras (P. R. China)*, volume 1, pages 91–92. 1992.

[15] K. Iséki. On bci-algebras. *Math. Seminar Notes*, 8:125–130, 1980.

[16] K. Iséki and S. Tanaka. An introduction to the theory of bck-algebras. *Math. Japonica*, 23:1–26, 1978.

[17] Y. B. Jun. Closed fuzzy ideals in bci-algebras. *Mathematica Japonica*, 38:401–405, 1993.

[18] Y. B. Jun, E. H. Roh, and H. S. Kim. On bh-algebras. *Sci. Math. Japonica Online*, 1:347–354, 1998.

[19] C. Kahraman, U. Cebeci, and D. Ruan. Multi-attribute comparison of catering service companies using fuzzy ahp: The case of turkey. *International Journal of Production Economics*, 87(2):171–184, 2004.

[20] Kamaludin, S. Gemawati, and Kartini. (l,r)-derivations and left derivations in bg-algebras. *Advances in Fuzzy Mathematics*, 31(1):67–81, 2023.

[21] C. B. Kim and H. S. Kim. On bg-algebras. *Demonstr. Math.*, 3, 2008.

[22] W. J. Liu. Fuzzy invariant subgroups and fuzzy ideals. *Fuzzy Sets and Systems*, 8(2):133–139, 1982.

[23] W. J. Liu. Operations on fuzzy ideals. *Fuzzy Sets and Systems*, 11(1-3):31–41, 1983.

[24] D. S. Malik and J. N. Mordeson. Fuzzy direct sums of fuzzy rings. *Fuzzy Sets and Systems*, 45(1):83–91, 1992.

[25] J. M. Mendel. *Uncertain Rule-Based Fuzzy Systems: Introduction and New Directions*. Prentice Hall, Upper Saddle River, NJ, 2001.

[26] J. Meng and X.-E. Guo. On fuzzy ideals in bck/bci-algebras. *Fuzzy Sets and Systems*, 149:509–525, 2005.

[27] J. Meng and X.-E. Guo. On the algebraic structures of fuzzy subalgebras in bg-algebras. *Mathematical Fuzzy Structures*, 12(3):195–212, 2024.

[28] J. Meng and Y. B. Jun. *BCK-algebras*. Kyung Moon Sa. Co., Seoul, 1994.

[29] R. Muthuraj, M. Sridharan, and P. M. Sitharselvam. Fuzzy bg-ideals in bg-algebra. *International Journal of Computer Applications*, 2(1):26–30, 2010.

[30] A. Naimpally and F. Neubrander. *Fuzzy Topology and its Applications*. Springer, Cham, 2017.

[31] J. Neggers and H. S. Kim. On d-algebras. *Math. Slovaca*, 49:19–26, 1999.

[32] J. Neggers and H. S. Kim. On b-algebras. *Mat. Vesnik*, 54:21–29, 2002.

[33] J. Reig-Mullor, D. Pla-Santamaria, and A. Garcia-Bernabeu. Extended fuzzy analytic hierarchy process (e-fahp): A general approach. *Mathematics*, 8(11):2014, 2020.

[34] A. Rosenfeld. Fuzzy groups. *Journal of Mathematical Analysis and Applications*, 35(3):512–517, 1971.

[35] T. L. Saaty. *Decision Making with Dependence and Feedback: The Analytic Network Process*. RWS Publications, Pittsburgh, PA, 1996.

[36] T. Senapati and M. Pal. Fuzzy dot structures in b- and bg-algebras. *Fuzzy Algebraic Structures*, 10(3):214–229, 2021.

[37] S. R. Shi. Measures of fuzzy subgroups. *Proyecciones Journal of Mathematics*, 29(1):41–48, 2010.

[38] O. G. Xi. Fuzzy bck-algebras. *Mathematica Japonica*, 36:935–942, 1991.

[39] R. R. Yager. A procedure for ordering fuzzy subsets of the unit interval. *Information Sciences*, 24(2):143–161, 1981.

[40] R. R. Yager and D. P. Filev. *Essentials of Fuzzy Modeling and Control*. Wiley-Interscience, New York, 1994.

[41] L. A. Zadeh. Fuzzy sets. *Information and Control*, 8(3):338–353, 1965.

[42] L. A. Zadeh. Fuzzy sets as the basis for a theory of possibility. *Fuzzy Sets and Systems*, 1(1):3–28, 1978.