Soliton Solutions of Nonlinear Fractional Differential Fquations via Functional Variable Method

Patanjali Sharma
Department of Education in Science and Mathematics
Regional Institute of Education
Ajmer 305004 INDIA

ABSTRACT

In this paper, the functional variable method is utilized to derive analytical solutions for the (2 + 1)-dimensional time-fractional Zoomeron equation and the space-time fractional modified regularized long-wave equation, based on the Jumarie's modified Riemann-Liouville derivative. The given equations are transformed into nonlinear ordinary differential equations of integer order, which are then solved using the proposed functional variable method, a novel analytical approach. Consequently, several exact solutions are successfully obtained. The results demonstrate that the proposed method is both efficient and easy to implement.

Keywords

(2 + 1)-dimensional time-fractional Zoomeron equation, fractional modified regularized long-wave equation, Jumarie's modified Riemann-Liouville derivative, Functional Variable Method.

1. INTRODUCTION

Fractional differential equations (FDEs) have drawn considerable interest in recent years, as they are increasingly employed to model complex nonlinear phenomena in fields such as physics, biology, mathematics, economics, engineering, and other scientific areas. Many real-world systems are formulated using FDEs, making the study of their exact solutions essential for scientific research.Fractional partial differential equations are a generalized form of nonlinear partial differential equations. They play a crucial role in understanding and analyzing nonlinear phenomena in applied sciences.

In recent decades, numerous researchers have successfully investigated the exact solutions and analytical approximations of nonlinear Fractional differential equations (FDEs). Many semi-analytical and analytical methods, such as modified extended tanh function method [1], modified Kudryashov method [2] and q-homotopy analysis transform method [10], the modified (G'/G)-expansion method [4], the improved Bernoulli sub-equation function method [5], the Kudryashov method [6], the sine-Gordon expansion method [23] [7, 13],the first integral method [9, 10], the extended trial equation method [22], the (G'/G, 1/G)-expansion method [34], the improved F-expansion method [35], the modified simple equation method [15, 16], the improved Bernoulli sub-equation function method [17], the Darboux transformation method [18, 19, 20], fractional subequation method [21], modified trial equation method [22], etc. have been employed for obtaining new exact solutions of the nonlinear partial diffferential equations including integer and fractional orders.

A powerful and effective method for finding exact solutions of nonlinear partial differential equations, known as the functional variable method, was proposed by Zerarka et al. in [23, 24]. More recently, Babajanov [25, 26, 27] applied this method to obtain soliton solutions for various differential equations. In [28], Jumarie introduced a modified Riemann?Liouville derivative. Using this type of fractional derivative along with some useful formulas, fractional differential equations can be transformed into integer-order differential equations through variable transformation.

This study aims to highlight the effectiveness of the functional variable method and the modified Riemann?Liouville derivative in solving nonlinear time-fractional differential equations to obtain exact solitary wave solutions, periodic wave solutions, and combined formal solutions.

The rest of this paper is structured as follows. In section 2, we proposed the basic idea of the method for finding exact travelling wave solutions of nonlinear time-fractional differential equations. In section 3, we established the exact travelling wave solution for the (2+1)-dimensional time fractional Zoomeron equation and the space-time fractional modified regularized long-wave equation. Finally, Sections 4 and 5 present the graphical representations of the equations and the conclusions, respectively.

2. JUMARIE'S MODIFIED RIEMANN-LIOUVILLE DERIVATIVE AND THE FUNCTIONAL VARIABLE METHOD

Jumarie's modified Riemann-Liouville derivative of order α is defined as [29]:

$$\mathfrak{D}_t^{\alpha} f(t) = \begin{cases} \frac{1}{\Gamma(-\alpha)} \int_0^t (t-\zeta)^{-\alpha-1}[f(\zeta)-f(0)]d\zeta, & \alpha < 0, \\ \frac{1}{\Gamma(-\alpha)} \frac{d}{dt} \int_0^t (t-\zeta)^{-\alpha-1}[f(\zeta)-f(0)]d\zeta, & 0 < \alpha < 1, \\ (f^{(n)}(t))^{(\alpha-n)}, & \text{n} \leq \alpha \leq \text{n+1, n} \geq 1 \end{cases}$$

where $f : R \to R, x \to f(x)$ denote a continuous (but not necessarily differentiable) function. We list some important properties for Jumarie's fractional derivative as

$$\mathfrak{D}_t^{\alpha} t^r = \frac{\Gamma(1+r)}{\Gamma(1+r-\alpha)} t^{r-\alpha}, r > 0,$$

$$\mathfrak{D}_t^{\alpha} c = 0, \quad c = constant,$$

$$\mathfrak{D}_t^{\alpha}[af(t) + bg(t)] = a\mathfrak{D}_t^{\alpha} f(t) + b\mathfrak{D}_t^{\alpha} g(t),$$

$$\mathfrak{D}_t^{\alpha}(f(t)g(t)) = g(t)\mathfrak{D}_t^{\alpha} f(t) + f(t)\mathfrak{D}_t^{\alpha} g(t),$$

$$\mathfrak{D}_t^{\alpha} f(g(t)) = f_g^{'}(g(t))\mathfrak{D}_t^{\alpha} g(t) = \mathfrak{D}_g^{\alpha} f(g(t))(g^{'}(t))^{\alpha}.$$

Drawing inspiration from the works of Lu [30] and Zerarka et al. [31], we present the functional variable method for obtaining exact solutions of nonlinear time-fractional differential equations, as described below.

Let us consider the time-fractional differential equation with independent variables $t, x, y, z, ...$ and a dependent variable u

$$\mathcal{P}(u, \mathfrak{D}_t^{\alpha} u, u_x, u_y, u_z, \mathfrak{D}_t^{2\alpha} u, u_{xy}, u_{yz}, u_{xz}, ...) = 0 \qquad (1)$$

where the subscript denotes partial derivative. Using the variable transformation

$$u(t, x, y, z, ...) = U(\zeta), \quad \xi = a_1 x + a_2 y + a_3 z + ... \pm \frac{\omega t^{\alpha}}{\Gamma(1+\alpha)}$$

where a_i and ω are constants to be determined later; the fractional differential equation (1) is reduced to an ordinary differential equation (ODE)

$$\mathcal{Q}(U, U_{\zeta}, U_{\zeta\zeta}, U_{\zeta\zeta\zeta}, ...) = 0 \qquad (2)$$

Then we make a transformation in which the unknown function U is considered as a functional variable in the form

$$U_{\zeta} = F(U) \qquad (3)$$

and some successive derivatives of U are

$$U_{\zeta\zeta} = \frac{1}{2}(F^2(U))^{'}$$
$$U_{\zeta\zeta\zeta} = \frac{1}{2}(F^2(U))^{''}\sqrt{F^2(U)} \qquad (4)$$
$$U_{\zeta\zeta\zeta\zeta} = \frac{1}{2}\left[(F^2(U))^{'''}F^2 + \frac{1}{2}(F^2(U))^{''}(F^2(U))^{'}\right]$$

and so on, where $' = d/dU$

Substituting (4) into (2), we reduce the ODE (2) in terms of U, F and its derivatives as

$$\mathcal{R}(U, F, F^{'}, F^{''}, F^{'''}, ...) = 0 \qquad (5)$$

Equation (5) is particularly important because it admits analytical solutions for a large class of nonlinear wave-type equations. After integration, eq. (5) provides the expression for F, and this together with eq. (3) give relevant solutions to the original problem. In order to illustrate how the method works, we examine some examples treated by other approaches. This is discussed in the following section.

3. APPLICATIONS

3.1 The (2+1)-dimensional time fractional Zoomeron equation

In this subsection, we employ the Functional Variable Method(FVM) to obtain general exact solutions of the time fractional Zoomeron equation, which has been proposed as follows

$$\mathfrak{D}_{tt}^{2\alpha}\left(\frac{u_{xy}}{u}\right) - \left(\frac{u_{xy}}{u}\right)_{xx} + 2\mathfrak{D}_t^{\alpha}\left(u^2\right)_x = 0, \quad t > 0, 0 < \alpha \leq 1 \qquad (6)$$

where $u = u(x, y, t)$ is the amplitude of the relevant wave mode. Equation 6 plays a significant role in describing the evolution of a single scalar field and serves as a useful model for showcasing novel phenomena associated with boomerons and trappons [32]. This equation was initially introduced by Calogero and Degasperis [33]. In the past, many researchers have studied both the integer and fractional forms of the Zoomeron equation [34, 35, 36, 37, 38, 39].

Let us consider the travelling wave solutions of Eq. (6), and we perform the transformation:

$$u(x, y, t) = U(\zeta), \quad \zeta = ax + by - \frac{\omega t^{\alpha}}{\Gamma(1+\alpha)} \qquad (7)$$

where a, b and ω are constants.

Substituting transformation (7) with (1) into Eq.(6). Then Eq. (6) can be reduced to an ODE in the following form:

$$ab\omega^2\left(\frac{U^{''}}{U}\right)^{''} - a^3 b\left(\frac{U^{''}}{U}\right)^{''} - 2a\omega(U^2)^{''} = 0 \qquad (8)$$

Integrating Eq. (8) twice with respect to ζ, we get

$$ab(\omega^2 - a^2)U^{''} - 2a\omega U^3 - kU = 0 \qquad (9)$$

where, primes denote differentiation with respect to ζ and k is a non zero constant of integration, while the second constant of integration is vanishing.

Substituting eq. 4 into eq. 9, we get

$$\frac{ab(\omega^2 - a^2)}{2}(F^2(U))^{'} = kU + 2a\omega U^3 \qquad (10)$$

Integrating the Eq. 10 with respect to U with zero constants of integration, we have

$$F(U) = \pm\sqrt{\frac{\omega}{b(\omega^2 - a^2)}}\, U\sqrt{\frac{k}{a\omega} + U^2} \qquad (11)$$

From eq. 4 and 11 we deduce that

$$\int \frac{dU}{U\sqrt{\frac{k}{a\omega} + U^2}} = \pm\sqrt{\frac{\omega}{b(\omega^2 - a^2)}}\,(\zeta + \zeta_0) \qquad (12)$$

where ζ_0 is a constant of integration. After integrating 12, we have the following exact solutions, for $\frac{\omega}{b(\omega^2 - a^2)} > 0$:

$$u_{1,1}(x,t) = \pm\sqrt{\frac{k}{a\omega}}\, csch\left[\sqrt{\frac{k}{ab(\omega^2 - a^2)}}\left(ax + by - \frac{\omega t^{\alpha}}{\Gamma(1+\alpha)} + \zeta_0\right)\right] \qquad (13)$$

$$u_{1,2}(x,t) = \pm\sqrt{\frac{-k}{a\omega}}\,sech\left[\sqrt{\frac{k}{ab(a^2-\omega^2)}}\left(ax+by-\frac{\omega t^\alpha}{\Gamma(1+\alpha)}+\zeta_0\right)\right] \quad (14)$$

For $\frac{\omega}{b(\omega^2-a^2)} < 0$, we obtain periodic solutions as follows:

$$u_{1,3}(x,t) = \pm\sqrt{\frac{k}{a\omega}}\,csc\left[\sqrt{\frac{k}{ab(a^2-\omega^2)}}\left(ax+by-\frac{\omega t^\alpha}{\Gamma(1+\alpha)}+\zeta_0\right)\right] \quad (15)$$

$$u_{1,4}(x,t) = \pm\sqrt{\frac{-k}{a\omega}}\,sec\left[\sqrt{\frac{k}{ab(a^2-\omega^2)}}\left(ax+by-\frac{\omega t^\alpha}{\Gamma(1+\alpha)}+\zeta_0\right)\right] \quad (16)$$

3.2 The space-time fractional modified regularized long-wave equation

We next consider the space-time fractional modified regularized long-wave equation (mRLW) [40]

$$\mathfrak{D}_t^\alpha u + a\mathfrak{D}_x^\alpha u + bu^2\mathfrak{D}_x^\alpha u - c\mathfrak{D}_t^\alpha\mathfrak{D}_x^{2\alpha} u = 0,\ 0<\alpha\le 1 \quad (17)$$

where a, b, c are arbitrary constants. The regularized long wave model is a fundamental and significant model in nonlinear science, first introduced by Peregrine [41]. This model proves to be valuable in explaining various nonlinear phenomena across different scientific and engineering fields, including pressure waves in liquids, gas bubbles, ion-acoustic and hydrodynamic waves in plasma, phonon packets in nonlinear crystals, and longitudinal dispersive waves in elastic rods, among others.
Let us introduce the following transformations:

$$u(x,t) = U(\zeta),\ \zeta = \frac{x^\alpha}{\Gamma(1+\alpha)} + \omega\frac{t^\alpha}{\Gamma(1+\alpha)} \quad (18)$$

Here ω is a non-zero constant. Substituting 18 into 17, we get the following ODE:

$$(\omega+a)U' + bU^2U' - c\omega U''' = 0 \quad (19)$$

Further, integrating 19 with respect to ζ, we get

$$(\omega+a)U + \frac{b}{3}U^3 - c\omega U'' = 0 \quad (20)$$

Substituting eq. 4 into eq. 20, we get

$$\frac{c\omega}{2}(F^2(U))' = (\omega+a)U + \frac{b}{3}U^3 \quad (21)$$

Integrating the Eq. 21 with respect to U with zero constants of integration, we have

$$F(U) = \pm\sqrt{\frac{b}{6c\omega}}\,U\sqrt{\frac{6(\omega+a)}{b}+U^2} \quad (22)$$

From eq. 4 and 22 we deduce that

$$\int \frac{dU}{U\sqrt{\frac{6(\omega+a)}{b}+U^2}} = \pm\sqrt{\frac{b}{6c\omega}}\,(\zeta+\zeta_0) \quad (23)$$

where ζ_0 is a constant of integration. After integrating 23, we have the following exact solutions, for $\frac{b}{c\omega} > 0$:

$$u_{2,1}(x,t) = \pm\sqrt{\frac{6(\omega+a)}{b}}\,csch\left[\sqrt{\frac{(\omega+a)}{c\omega}}\left(\frac{x^\alpha}{\Gamma(1+\alpha)}+\omega\frac{t^\alpha}{\Gamma(1+\alpha)}+\zeta\right.\right. \quad (24)$$

$$u_{2,2}(x,t) = \pm\sqrt{\frac{-6(\omega+a)}{b}}\,sech\left[\sqrt{\frac{(\omega+a)}{c\omega}}\left(\frac{x^\alpha}{\Gamma(1+\alpha)}+\omega\frac{t^\alpha}{\Gamma(1+\alpha)}+\right.\right. \quad (25)$$

For $\frac{b}{c\omega} < 0$, we obtain periodic solutions as follows:

$$u_{2,3}(x,t) = \pm\sqrt{\frac{6(\omega+a)}{b}}\,csc\left[\sqrt{\frac{-(\omega+a)}{c\omega}}\left(\frac{x^\alpha}{\Gamma(1+\alpha)}+\omega\frac{t^\alpha}{\Gamma(1+\alpha)}+\right.\right. \quad (26)$$

$$u_{2,4}(x,t) = \pm\sqrt{\frac{-6(\omega+a)}{b}}\,sec\left[\sqrt{\frac{-(\omega+a)}{c\omega}}\left(\frac{x^\alpha}{\Gamma(1+\alpha)}+\omega\frac{t^\alpha}{\Gamma(1+\alpha)}-\right.\right. \quad (27)$$

4. GRAPHICAL REPRESENTATION

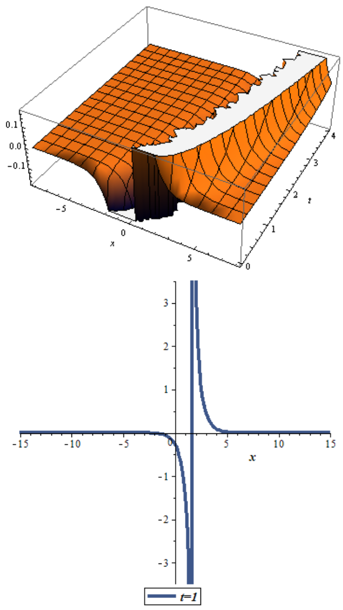
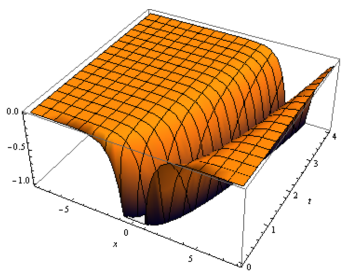


Fig.1: 3D and 2D plot of $u_{1,1}(x,t)$ given in Eq. 13 with $a=1, b=1, \omega=2, \alpha=0.5, k=-8, y=0, \zeta_0=0$.



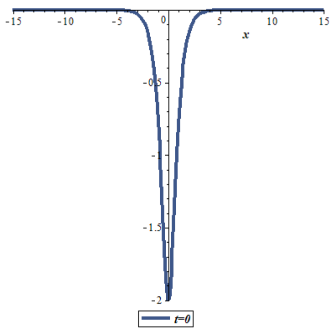


Fig.2: 3D and 2D plot of $u_{1,2}(x,t)$ given in Eq. 14 with$a=1,b=1,\omega=2,\alpha=0.5,k=-8,y=0,\zeta_0=0.$

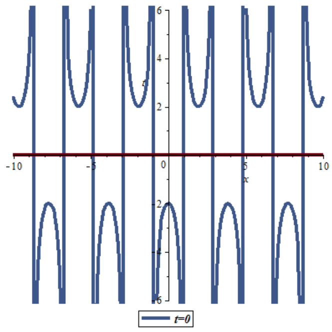


Fig.4: 3D and 2D plot of $u_{1,4}(x,t)$ given in Eq. 16 with $a=1,b=1,\omega=2,\alpha=0.5,k=-8,y=0,\zeta_0=0.$

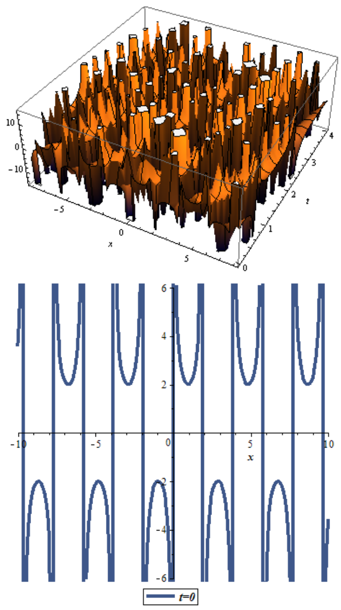


Fig.3: 3D and 2D plot of $u_{1,3}(x,t)$ given in Eq. 15 with $a=1,b=1,\omega=2,\alpha=0.5,k=-8,y=0,\zeta_0=0.$

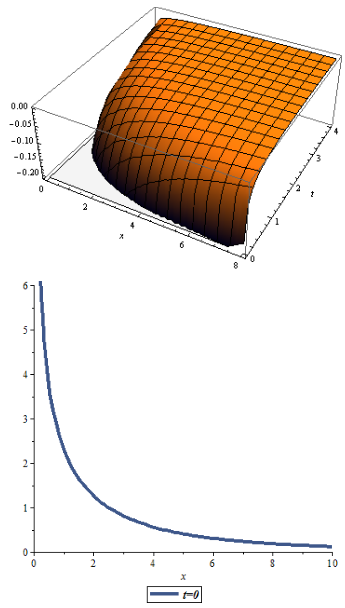
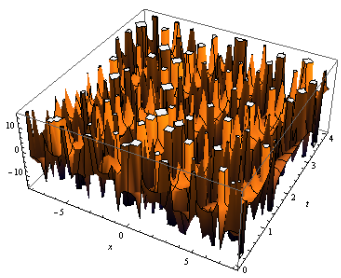
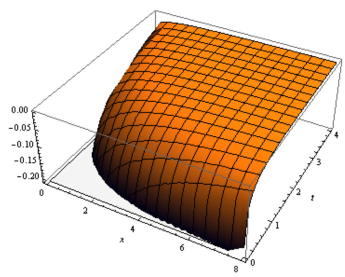


Fig.5: 3D and 2D plot of $u_{2,1}(x,t)$ given in Eq. 24 with $a=b=c=1,\omega=2,\alpha=0.5,\zeta_0=0.$





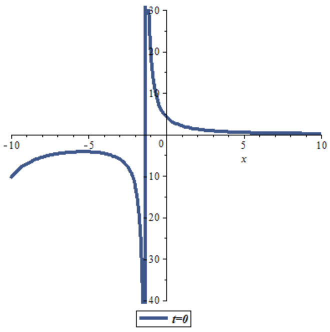


Fig.6: 3D and 2D plot of $u_{2,2}(x,t)$ given in Eq. 25 with $a = b = c = 1, \omega = 2, \alpha = 0.5, \zeta_0 = 0$.

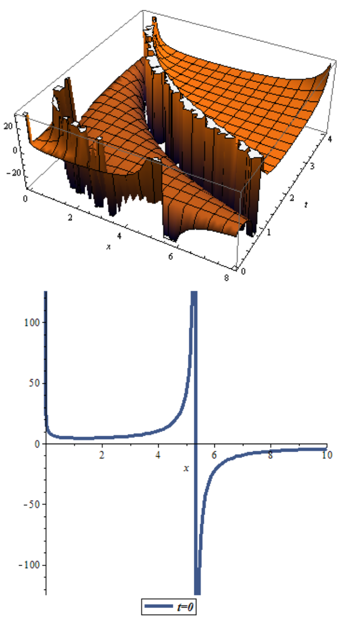
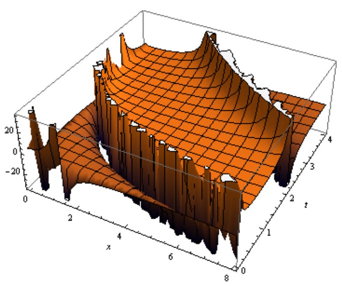


Fig.7: 3D and 2D plot of $u_{2,3}(x,t)$ given in Eq. 26 with $a = b = c = 1, \omega = 2, \alpha = 0.5, \zeta_0 = 0$.



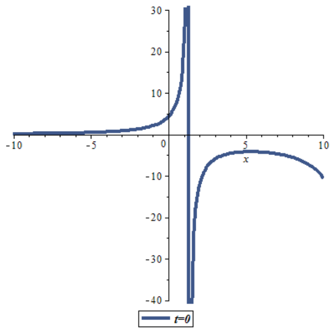


Fig.8: 3D and 2D plot of $u_{2,4}(x,t)$ given in Eq. 27 with $a = b = c = 1, \omega = 2, \alpha = 0.5, \zeta_0 = 0$.

We have provided graphs of solitary waves for equations 6 and 17, created by selecting suitable values for the relevant parameters, to better understand the mechanisms behind the original physical phenomena. Graphical representations are a powerful tool for communication, effectively illustrating the solutions to these problems. These graphs are displayed in Fig.1 to Fig.8. Solitary and periodic wave solutions are significant types of solutions for nonlinear partial differential equations, as many such equations exhibit various solitary wave solutions. Solitons, a specific class of solutions to nonlinear partial differential equations with weak linearity, are frequently used to model physical systems. The existence of periodic traveling waves generally depends on the parameter values in the mathematical equations, with these parameters affecting both amplitude and velocity. A soliton is a self-sustaining wave packet that retains its shape while traveling at a constant velocity.

5. CONCLUSION

In this paper, the functional variable method and the modified Riemann?Liouville derivative are introduced for solving the (2 + 1)-dimensional time-fractional Zoomeron equation and the space-time fractional modified regularized long-wave equation. It is anticipated that the solutions obtained in this study will be valuable for further exploration of complex nonlinear physical phenomena. This method provides a promising approach for solving a wide range of fractional partial differential equations and serves as a reliable technique for handling nonlinear fractional differential equations.
Additionally, this method is straightforward, concise, and especially well-suited for computer implementation. The algebraic complexities and extensive calculations were effectively managed using the symbolic computation software Mathematica. As a result, this approach can be extended to tackle nonlinear problems in soliton theory and other related areas.

6. REFERENCES

[1] A. A. Mamun, S. N. Ananna, T. An, N. H. M. Shahen, M. Asaduzzaman *Dynamical behaviour of travelling wave solutions to the conformable time-fractional modified Liouville and mRLW equations in water wave mechanics* Heliyon 7(8) (2021) e07704.

[2] S. Sirisubtawee, S. Koonprasert, S. Sungnul, T. Leekparn *Exact traveling wave solutions of the space-time fractional complex Ginzburg-Landau equation and the space-time fractional Phi-four equation using reliable methods* Adv. Diff. Equ. 2019 (2019) 219.

[3] R. Shah, Y. Alkhezi, K. Alhamad *An analytical approach to solve the fractional Benney equation using the q-homotopy analysis transform method* Symmetry 15(3) (2023) 669 .

[4] U. Ali, H. Ahmad, J. Baili, T. Botmart, M.A. Aldahlan *Exact analytical wave solutions for space-time variable-order fractional modified equal width equation* Results Phys. 33 (2022) 105216

[5] U. Zaman, M.A. Arefin, M.A. Akbar, M.H. Uddin *Solitary wave solution to the space-time fractional modified equal width equation in plasma and optical fiber systems* Results Phys. 52 (2023) 106903

[6] K. Raslan, K.K. Ali, M.A. Shallal *The modified extended tanh method with the Riccati equation for solving the space-time fractional ew and mew equations* Chaos Solitons Fract. 103 (2017) 404?409

[7] M.A. Khatun, M.A. Arefin, M.A. Akbar, M.H. Uddin *Numerous explicit soliton solutions to the fractional simplified Camassa-Holm equation through two reliable techniques* Ain Shams Eng. J. 14 (2023), 102214.

[8] H M Baskonus *New acoustic wave behaviors to the Davey-Stewartson equation with power-law nonlinearity arising in fluid dynamics* Nonlinear Dyn 86 (2016) 77?83.

[9] K R Raslan *The first integral method for solving some important nonlinear partial differential equations* Nonlinear Dyn 53(4) (2008) 959?979.

[10] P Sharma, O Y Kushel *The First Integral Method for Huxley Equation* International Journal of Nonlinear Science 10(1) (2010) 46-52.

[11] S T Demiray, Y Pandir, H Bulut *All exact travelling wave solutions of Hirota equation and Hirota-Maccari system* Optik 127(4) (2016) 1848?1859.

[12] M Kaplan, A Bekir, M N Ozer *Solving nonlinear evolution equation system using two different methods* Open Phys 13 (2015) 383?388.

[13] H M Baskonus *New acoustic wave behaviors to the Davey-Stewartson equation with power-law nonlinearity arising in fluid dynamics* Nonlinear Dyn 86 (2016) 77?83.

[14] M A Akbar, N M Ali *The improved F-expansion method with Riccati equation and its application in mathematical physics* Cogent Math Stat 41 (2017) 282?577.

[15] M A Kayum, S Ara, H K Barman, M A Akbar *Soliton solutions to voltage analysis in nonlinear electrical transmission lines and electric signals in telegraph lines* Results Phys 18 (2020) 103269.

[16] M A Kayum, M A Akbar, M S Osman *Competent closed form soliton solutions to the nonlinear transmission and the low pass electrical transmission lines* Eur Phys J Plus 135 (2020) 575.

[17] M E Islam, H K Barman, M A Akbar *Search for interactions of phenomena described by the coupled Higgs field equations through analytical solutions* Opt Quant Electron 52 (2020) 468.

[18] W Y Guan, B Q Li *Mixed structures of optical breather and rogue wave for a variable coefficient inhomogeneous fiber system* Opt Quant Electron 51 (2019) 352.

[19] Y L Ma *Interaction and energy transition between the breather and rogue wave for a generalized nonlinear Schrodinger system with two higher-order dispersion operators in optical fibers* Nonlinear Dyn 97 (2019) 95?105.

[20] B Q Li, Y L Ma Extended generalized Darboux transformation to hybrid rogue wave and breather solutions for a nonlinear Schrodinger equation Appl Math Comput 386 (2020) 125469.

[21] F. Meng, Q. Feng *Exact solutions with variable coefficient function forms for conformable fractional partial differential equations by an auxiliary equation method* Advances in Mathematical Physics vol. 2018, Article ID 4596506, 8 pages, 2018.

[22] S. T. Demiray, S. Duman *MTEM to the (2+1)-dimensional ZK equation and Chafee-Infante equation* Advanced Mathematical Models and Applications 6(1) (2021) 63?70.

[23] A Zerarka, S Ouamane, A Attaf *On the functional variable method for finding exact solutions to a class of wave equations*, Appl. Math. and Comput. 217 (2010) 2897-2904

[24] A Zerarka, S Ouamane *Application of the functional variable method to a class of nonlinear wave equations*, World Journal of Modelling and Simulation, 6(2) (2010) 150-160

[25] B Babajanov, F Abdikarimov *The Application of the Functional Variable Method for Solving the Loaded Non-linear Evaluation Equations*, Frontiers in Applied Mathematics and Statistics, 6 (2022) 912674

[26] B Babajanov, F Abdikarimov *Soliton and Periodic Wave Solutions of the Nonlinear Loaded (3+1)-Dimensional Version of the Benjamin-Ono Equation by Functional Variable Method*, Journal of Nonlinear Modeling and Analysis, 5(4) (2023) 782?789

[27] B Babajanov, F Abdikarimov *New soliton solutions of the Burgers equation with additional time-dependent variable coefficient*, WSEAS TRANSACTIONS on FLUID MECHANICS, 19 (2024) 59-63

[28] G. Jumarie *Modified Riemann?Liouville derivative and fractional Taylor series of nondifferentiable functions further results* Comput. Math. Appl. 51 (2006) 1367?1376.

[29] G. Jumarie *Table of some basic fractional calculus formulae derived from a modified Riemann?Liouvillie derivative for non differentiable functions*, Appl. Math. Lett. 22 (2009) 378?385.

[30] B Lu *The first integral method for some time fractional differential equations* J. Math. Anal. Appl. 395(2) (2012) 684-693.

[31] A Zerarka, S Ouamane and A Attaf *On the functional variable method for finding exact solutions to a class of wave equations* Appl. Math. Comput. 217(7) (2010) 2897-2904

[32] E. Tala-Tebue, Z.I. Djoufack, A. Djimeli-Tsajio, A. Kenfack-Jiotsa *Solitons and other solutions of the nonlinear fractional Zoomeron equation* Chin. J. Phys. 56(3) (2018) 1232?1246.

[33] F. Calogero, A. Degasperis *Nonlinear evolution equations solvable by the inverse spectral transform I. II* Nuovo Cimento B 32 (1976) 201?242.

[34] M. Kaplan, A. Bekir, M.N. Ozer *A simple technique for constructing exact solutions to nonlinear differential equations with conformable fractional derivative*, Opt. Quant. Electron. 49 (8) (2017) 266.

[35] A. Akbulut, M. Kaplan *Auxiliary equation method for time-fractional differential equations with conformable derivative*, Comput. Math. Appl. 75 (3) (2018) 876?882.

[36] E. Aksoy, A.C. vikel, A. Bekir *Soliton solutions of (2+1)-dimensional time-fractional Zoomeron equation*, Optik 127 (17) (2016) 6933?6942.

[37] O. Guner, A. Bekir *Bright and dark soliton solutions for some nonlinear fractional differential equations*, Chin. Phys. B 25 (3) (2016) 030203.

[38] F Batool, G Akram,M Sadaf, U Mehmood *Dynamics Investigation and Solitons Formation for (2+1)-Dimensional Zoomeron Equation and Foam Drainage Equation* Journal of Nonlinear Mathematical Physics 30 (2023) 628?645.

[39] M Alshammari, K Moaddy, M Naeem, Z Alsheekhhussain et al. *Solitary and Periodic Wave Solutions of Fractional Zoomeron Equation* fractal and fractional 8(4) (2024) 222.

[40] S Hossain, M.M. Roshid, M Uddin et al. *Abundant time-wavering solutions of a modified regularized long wave model using the EMSE technique* Partial Differential Equations in Applied Mathematics 8 (2023) 100551.

[41] D.H. Peregrine *Calculations of the development of an undular bore* J Fluid Mech 25(2) (1966) 321-330.