Application of Linear Space on Trapezoidal Intuitionistic Fuzzy Numbers based on (α, β)-cut Method

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ABSTRACT
Trapezoidal Fuzzy number (TrFN) and Trapezoidal intuitionistic Fuzzy number (TrIFN) play an important role in various field. TrINF is an extension of standard fuzzy numbers that handles uncertainty by using both a membership function and non membership function with a trapezoidal shape. .In this research paper we discussed some elementary operations on proposed trapezoidal fuzzy numbers (TrFNs).We also compare the trapezoidal fuzzy number and intuitionistic Trapezoidal fuzzy number as in the formation of Linear space. Membership and non- membership function are calculated in terms of (α, β)-cut method of intutionistic trapezoidal fuzzy numbers. Numerical example and application of an electric network model of dark room is provided here.

Keywords
Trapezoidal Fuzzy Numbers, Trapezoidal Intuitionistic Fuzzy Numbers, Trapezoidal Fuzzy Linear space, Membership and non- membership function.

1. INTRODUCTION
Fuzzy set theory was introduced by Zadeh in 1965[15].It has been researched widely in many fields and various theories and generalization have been appeared in different directions. Intuitionistic Fuzzy set (IFS)is one of the generalization of fuzzy sets theory[.The concept of intuitionistic fuzzy set (IFS) is first introduced by Atanassov[1], [2]and [3].IFS can be viewed as an alternative appropriate approach to define a fuzzy set in case where available information is not sufficient for the definition of an imprecise concept by means of a conventional fuzzy set.Out of several higher-order fuzzy set , IFSs have been found to be compatible to deal with vagueness. In fuzzy sets only the degree of acceptance is considered but IFS is characterized by a membership function and non-membership function so that the sum of both values is less than one [4].Two theorems for intuitionistic fuzzy sets are explained in [5].There are many studies about the inter-valued intuitionistic fuzzy sets and systems[6] and Correlation of intuitionistic fuzzy sets is explained in [10].The relation between the fuzzy operators are defined over the intuitionistic fuzzy sets [13] and Cartesian product over intuitionistic fuzzy sets is explained in[14].Introduction of positive and non-negative fuzzy numbers[11]and distance measures evaluation between intuitionistic fuzzy multisets [9].The trapezoidal fuzzy number for solving fuzzy linear programming problem using Alpha – cut method [**7**].Application of reliability are discussed [**12].**

In this paper we discussed introduction of trapezoidal fuzzy linear space and it is proved that set of all intuitionistic triangular fuzzy numbers does not form a linear space .Also the membership and non-membership functions are discussed. Section 2 deals with some basic definition of fuzzy number and intuitionistic fuzzy number. In section 3, New Modified method of Trapezoidal fuzzy linear space and its properties are discussed. In section 4, Arithmetic operations of triangular intuitionistic fuzzy number based on (α, β)-cut method are discussed. In section 5, Numerical example and application of an electric network model of dark room is presented. Concluding remarks are given in section 6.

2. BASIC CONCEPT OF FUZZY SET AND INTUITIONISTIC FUZZY SET
The fuzzy set theory was first introduced by Zadeh in 1965 [15].Let X be universe of discourse defined by $X = \{x_1, x_2, \dots, x_n\}$.The grade of membership of element $x_i \in X$ in the fuzzy set is represented by real value between 0 and 1.It does indicate the evidencing for $x_i \in X$,but does not indicate the evidence against $x_i \in X$. Atanassov [3],[5] introduced the concept of IFS,and pointed out that this single value combines the evidence for $x_i \in X$ and the evidence against $x_i \in X$.An IFS $\tilde{A}$ in X is characterized by two functions expressing the degree of membership function $\mu_{\tilde{A}}(x)$ and the degree of non-membership function $v_{\tilde{A}}(x)$ respectively.Here $\mu_{\tilde{A}}(x)$ and $v_{\tilde{A}}(x)$ are associated with each point in X, a real number in[0,1] with the values of $\mu_{\tilde{A}}(x)$ and $v_{\tilde{A}}(x)$ at X representing the grade of membership and non- membership of x in $\bar{A}$.Thus closure the value of $\mu_{\tilde{A}}(x)$ to unity and the value of $v_{\tilde{A}}(x)$ to zero;higher the grade of membership and lower the grade of non-membership of x.When $\tilde{A}$ is an ordinary set its membership function and non-membership function can take on only two values zero and one. If $\mu_{\tilde{A}}(x) = 1$ and $v_{\tilde{A}}(x) = 0$ the element x does not belongs to $\tilde{A}$,similarly if $\mu_{\tilde{A}}(x) = 0$ and $v_{\tilde{A}}(x) = 1$ the element x does not belong to $\tilde{A}$. An IFS becomes a fuzzy set $\tilde{A}$ when $v_{\tilde{A}}(x) = 0$ and $\mu_{\tilde{A}}(x) \epsilon [0,1] \forall x \in \tilde{A}$.

.1 Definition: Trapezoidal Fuzzy number [7]
It is a fuzzy number represented with four vertices as follows: $\widetilde{A}$=(a, b, c, d) this representation is interpreted as membership functions and hold the following conditions

i. a to b is strictly increasing function.
ii. 1, where b=c
iii. c to d is strictly decreasing function.
iv. $a \le b \le c \le d$

$$\mu_{\tilde{A}}(x) = \begin{cases} 0 \; for \; x<a \\ \frac{x-a}{b-a} \quad for \; a \le x \le b \\ 1 \quad for \; b \le x \le c \\ \frac{d-x}{d-c} \quad for \; c \le x \le d \\ 0 \quad for x>d \end{cases}$$

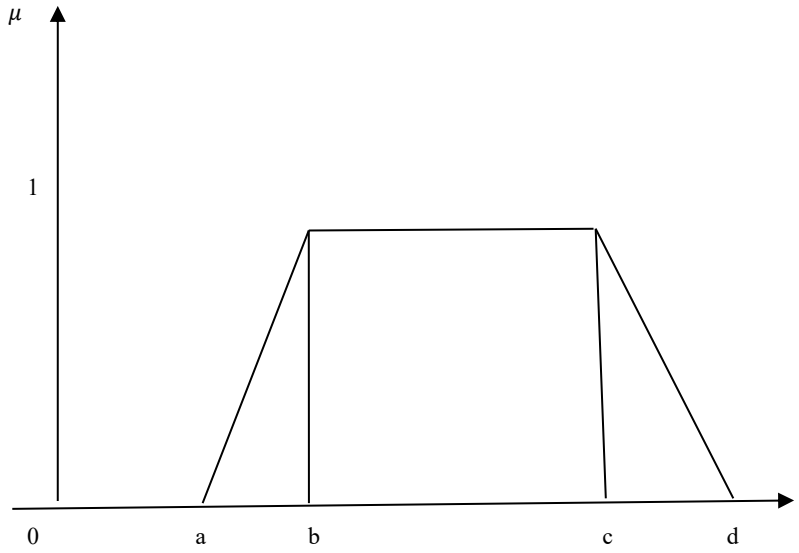

Fig 1: Trapezoidal Fuzzy Number $\tilde{A}$ =(a, b, c, d)

α- cut of a TrFN for this shape is denoted as A_α and is defined as

A_α = [(b –a) α + a, - (d-c) α +d] $\forall\alpha \in [0,1]$. When a=b the trapezoidal fuzzy number coincides with triangular one

2.2 Definition

Positive Trapezoidal Fuzzy Number: A positive trapezoidal fuzzy number is a fuzzy number $\tilde{A}$ is defined as

$\tilde{A} = (a_1, a_2, a_3, a_4)$ where all $a_i > 0, \forall i = 1,2,3,4$

2.3 Definition

Negative Trapezoidal Fuzzy Number: A negative trapezoidal fuzzy number is a fuzzy number $\tilde{A}$ is defined as

$\tilde{A} = (a_1, a_2, a_3, a_4)$ where all $a_i < 0, \forall i = 1,2,3,4$.

Remark: A negative trapezoidal fuzzy number can be written as the negative multiplication of a positive trapezoidal fuzzy number.

2.4 Definition: (α, β) – Level Intervals or (α, β) –Cuts. A set of (α, β) –, generated by IFS $\tilde{A}$, where $(\alpha, \beta) \in [0, 1]$ are fixed numbers such that $\alpha + \beta \leq 1$ is defined as

$$\tilde{A}_{\alpha,\beta} = \begin{cases} (x, \mu_{\tilde{A}}(x), v_{\tilde{A}}(x)) : x \in X \\ \mu_{\tilde{A}}(x) \geq \alpha, v_{\tilde{A}}(x) \leq \beta : \alpha, \beta \in [0,1] \end{cases}$$

(α, β) – Level Intervals or (α, β) –Cut, denoted by $\tilde{A}_{\alpha,\beta}$ is defined as the crisp set of elements x which is belong to $\tilde{A}$ at least to the degree α and which at most to the degree β.

2.5 Definition: Trapezoidal Intuitionistic Fuzzy Number

A Trapezoidal Intuitionistic Fuzzy Number (TrIFN) $\tilde{A}$ is an intuitionistic fuzzy number in R with the following membership function $(\mu_{\tilde{A}}(x))$ and non –membership function $(v_{\tilde{A}}(x))$as

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{x-a_1}{a_2-a_1} & for a_1 \leq x \leq a_2 \\ 1 & for\ a_2 \leq x \leq a_3 \\ \frac{a_4-x}{a_4-a_3} & for\ a_3 \leq x \leq a_4 \\ 0 & otherwise \end{cases} \text{ and}$$

$$v_{\tilde{A}}(x) = \begin{cases} \frac{a'_1 - x}{a_2 - a'_1} & for a'_1 \leq x \leq a_2 \\ 0 & for\ a_2 \leq x \leq a_3 \\ \frac{x - a'_4}{a'_4 - a_3} & for\ a_3 \leq x \leq a'_4 \\ 1 & otherwise \end{cases},$$

Where $a'_1 \leq a_1 \leq a_2 \leq a_3 \leq a_4 \leq a'_4$ and $\mu_{\tilde{A}}(x), v_{\tilde{A}}(x) \leq 0.5$ for $\mu_{\tilde{A}}(x) = v_{\tilde{A}}(x)\ \forall\ x \in R$ and trapezoidal intuitionistic fuzzy number is denoted by $\tilde{A}_{TrIFN} = (a_1, a_2, a_3, a_4; a'_1, a_2, a_3, a'_4)$.

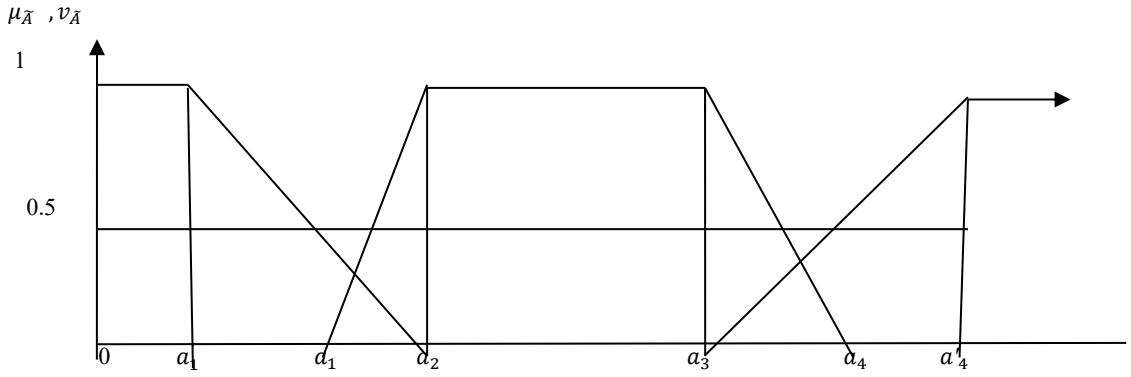

Fig. 2 Membership and non membership functions of TrIFN.

2.6 Definition: Positive Trapezoidal Intuitionistic Fuzzy Number

A Positive Trapezoidal Intuitionistic Fuzzy Number (TrIFN) $\tilde{A}$ is an intuitionistic fuzzy number in R with the membership function ($\mu_{\tilde{A}}(x)$) and non –membership function ($v_{\tilde{A}}(x)$)with all a_i`s are positive follows

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{x-a_1}{a_2-a_1} & for a_1 \le x \le a_2 \\ 1 & for\ a_2 \le x \le a_3 \\ \frac{a_4-x}{a_4-a_3} & for\ a_3 \le x \le a_4 \\ 0 & otherwise \end{cases},$$

and

$$v_{\tilde{A}}(x) = \begin{cases} \frac{a'_1 - x}{a_2 - a'_1} & for a'_1 \le x \le a_2 \\ 0 & for\ a_2 \le x \le a_3 \\ \frac{x - a'_4}{a'_4 - a_3} & for\ a_3 \le x \le a'_4 \\ 1 & otherwise \end{cases},$$

Where $a_1' \le a_1 \le a_2 \le a_3 \le a_4 \le a_4'$ and it is denoted by $\tilde{A}_{TrIFN} = (a_1, a_2, a_3, a_4; a_1' a_2, a_3, a'_4)$.

2.7 Definition: Negative Trapezoidal Intuitionistic Fuzzy Number

A Negative Trapezoidal Intuitionistic Fuzzy Number (TIFN) $\tilde{A}$ is an intuitionistic fuzzy number in R with the membership function and non –membership function as follows

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{x+a_4}{-a_3+a_4} & for\ -a_4 \le x \le -a_3 \\ 1 & for -a_3 \le x \le -a_2 \\ \frac{-a_1-x}{a_2-a_1} & for\ -a_2 \le x \le -a_1 \\ 0 & otherwise \end{cases},$$ and

$$v_{\tilde{A}}(x) = \begin{cases} \frac{a'_4 - x}{a_3 - a'_4} & for\ -a'_4 \le x \le -a_3 \\ 0 & for -\ a_3 \le x \le -a_2 \\ \frac{x - a'_1}{-a'_1 + a_2} & for -a_2 \le x \le -a'_1 \\ 1 & otherwise \end{cases},$$

Where $-a'_4 \le -a_4 \le -a_3 \le -a_2 \le -a_1 \le a_1'$ and it is denoted by $\tilde{A}_{TrIFN} = (-a_4 - a_3, -a_2, -a_1; -a_4' \le a_3 - a_2, -a_1')$.

2.8. Arithmetic operations of trapezoidal fuzzy numbers.

The following are the four operations that can be performed on TrFN: Let $\tilde{A} = (a_1, a_2, a_3, a_4)$ and $\tilde{B} = (b_1, b_2, b_3, b_4)$ then,

(i) Addition:

$\tilde{A}$ (+) $\tilde{B}$= (a_1, a_2, a_3, a_4) (+)(b_1, b_2, b_3, b_4)

=($a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4$)

(ii) Subtraction:

$\tilde{A}$ (-) $\tilde{B}$= (a_1, a_2, a_3, a_4) (-)(b_1, b_2, b_3, b_4)

= ($a_1 - b_1, a_2 - b_2, a_3 - b_3, a_4 - b_4$)

Also, 1-$\tilde{A}$= 1-(a_1, a_2, a_3, a_4)

= (1-$a_1, 1 - a_2, 1 - a_3, 1 - a_4$)

(iii) Multiplication:

$\tilde{A}$ (x) $\tilde{B}$= (a_1, a_2, a_3, a_4) (x)(b_1, b_2, b_3, b_4)

=($a_1b_1, , a_2b_2, a_3b_3, a_4b_4$)

Also, k$\tilde{A}$=k (a_1, a_2, a_3, a_4) = (k a_1ka_2,ka_3,k a_4)

where k is any real number.

(iv) Division:

$\tilde{A}(\div)\tilde{B}$ = (a_1, a_2, a_3, a_4) (÷)(b_1, b_2, b_3, b_4)

= $(\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, \frac{a_4}{b_4})$ where $\forall b_i \neq 0$

3. TRAPEZOIDAL FUZZY LINEAR SPACE.

In this section, we discussed the Linear Space in the trapezoidal fuzzy form.

Let F is an arbitrary non –empty set of trapezoidal fuzzy number on which two operations addition and multiplication by scalar are defined .If the following postulates are satisfied by all numbers of F then F is called a Trapezoidal fuzzy Linear space.

Taking $\tilde{A}, \tilde{B}, \tilde{C}$ are trapezoidal fuzzy numbers and k, m be any real constants then

3.1. Laws of addition.

(i) Closure:

$\tilde{A}$ (+) $\tilde{B}$= (a_1, a_2, a_3, a_4) +(b_1, b_2, b_3, b_4)

= ($a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4$) is again a triangular fuzzy number. So it is closed.

(ii) Associative:

[$\tilde{A}$ (+) $\tilde{B}$] (+)$\tilde{C}$= [a_1, a_2, a_3, a_4) + (b_1, b_2, b_3, b_4)] + (c_1, c_2, c_3, c_4)

=($a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4$)+($c_1, c_2, c_3, c_4$)

=($a_1 + b_1 + c_1, a_2 + b_2 + c_2, a_3 + b_3 + c_3, a_4 + b_4 + c_4$)

= (a_1, a_2, a_3, a_4) + ($b_1 + c_1, b_2 + c_2, b_3 + c_3, b_4 + c_4$))

= (a_1, a_2, a_3, a_4) +[(b_1, b_2, b_3, b_4)] + (c_1, c_2, c_3, c_4)]

= $\tilde{A}$ (+) $[\widetilde{B}$ (+)$\tilde{C}]$ are again a trapezoidl fuzzy number. Associativily is satisfied.

(iii) Identity:

0 (+) $\tilde{A}$ = (0, 0, 0, 0) + (a_1, a_2, a_3, a_4)

= ($0 + a_1, 0 + a_2, 0 + a_3, 0 + a_4$)

= ($a_1 + 0, a_2 + 0, a_3 + 0, a_4 + 0$)

= (a_1, a_2, a_3, a_4) + (0,0,0,0)

= $\tilde{A}$ + 0, 0 is called the identity element.

(iv) Inverse:

$\tilde{A}$ (+) $(-\tilde{A})$= (a_1, a_2, a_3, a_4) + ($-a_1, -a_2, -a_3, -a_4$)

= ($a_1 - a_1, a_2 - a_2, a_3 - a_3, a_4 - a_4$)

= (- $a_1 + a_1, -a_2 + a_2, -a_3 + a_3, -a_4 + a_4$)

= ($-a_1, -a_2, -a_3, -a_4$) + (a_1, a_2, a_3, a_4)

= $(-\tilde{A})$ (+) $\tilde{A}$

= (0, 0, 0, 0), $(-\tilde{A})$ is called the additive inverse of $\tilde{A}$.

(v) Commutative:

$\tilde{A}$ (+) $\tilde{B}$= (a_1, a_2, a_3, a_4) + (b_1, b_2, b_3, b_4)

= ($a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4$)

= ($b_1 + a_1, b_2 + a_2, b_3 + a_3, b_4 + a_4$)

= (b_1, b_2, b_3, b_4) + (a_1, a_2, a_3, a_4)

= (b_1, b_2, b_3, b_4) +(a_1, a_2, a_3, a_4)

= $\tilde{B}$ (+) $\tilde{A}$, Commutativity is satisfied.

3.2 Laws of scalar multiplication.

(i) k [$\tilde{A}$ (+) $\tilde{B}$]= k[(a_1, a_2, a_3, a_4) +(b_1, b_2, b_3, b_4)]

=k [($a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4$]

= ($ka_1 + kb_1, ka_2 + kb_2, ka_3 + kb_3, ka_4 + kb_4$)

= (k a_1, ka_2, ka_3, ka_4) +(kb_1, kb_2, kb_3, kb_4)

= k (a_1, a_2, a_3, a_4) + k(b_1, b_2, b_3, b_4)

= k $\tilde{A}$ (+) k $\tilde{B}$

(ii) (k + m)$\tilde{A}$ = (k+m) ($a_1, a_2, a_3, a_4; a_1', a_2, a_3, a_4'$)

= (($k + m)a_1, (k + m)a_2, (k + m)a_3, (k + m)a_4; (k + m)a_1', (k + m)a_2, (k + m)a_3, (k + m)a'_4$)

= (($ka_1 + ma_1), (ka_2 + ma_2)(ka_3 + ma_3), (ka_4 + ma_4'); (ka_1' + ma_1'), (ka_2 + ma_2), (ka_3 + ma_3), (ka_4' + ma_4')$)

= ($ka_1, ka_2, ka_3, ka_4; ka_1' ka_2, ka_3, ka_4'$)
+($ma_1, ma_2, ma_3, ma_4; ma_1', ma_2, ma_3, ma_4'$)

= k ($a_1, a_2, a_3, a_4; a_1' a_2, a_3, a_4'$)+m ($a_1, a_2, a_3, a_4; a_1' a_2, a_3, a_4'$).

= (k$\tilde{A}$+m$\tilde{A}$)

(iii) (k m)$\tilde{A}$ =k[m($a_1, a_2, a_3, a_4; a_1' a_2, a_3, a_4'$)

= k ($ma_1, ma_2, ma_3, ka_4; ma_1', ma_2, ma_3, ma_4'$)

= ($kma_1, kma_2, kma_3, kma_4; kma_1', kma_2, kma_3, kma_4'$)

= km($a_1, a_2, a_3, a_4; a_1' a_2, a_3', a_4'$)

=k(m$\tilde{A}$).

(v) 1.$\tilde{A}$ = 1.($a_1, a_2, a_3, a_4; a_1' a_2, a_3, a_4'$)

= ($1.a_1, 1.a_2, 1.a_3\ 1.a_4; 1.a_1', 1.a_2, 1.a_3 1.a_4'$)

=($a_1, a_2, a_3, a_4; a_1' a_2, a_3, a_4'$) = $\tilde{A}$

So, all the conditions of linear space are satisfied .Therefore, the set of all Trapezoidal Fuzzy numbers form a linear space called Trapezoidal fuzzy linear space.

3.3. Arithmetic operations of Trapezoidal Intuitionistic Fuzzy Number.

Let $\tilde{A}$ be Trapezoidal intuitionistic Fuzzy number, denoted as

$\tilde{A}_{TrIFN}$= ($a_1, a_2, a_3, a_4; a_1' a_2, a_3, a_4'$) where $a_1' \le a_1 \le a_2 \le a_3 \le a_4 \le a_4'$ and Negative Trapezoidal Intuitionistic Fuzzy number is defined as $\tilde{A}_{TrIFN}$= ($-a_4, -a_3, -a_2, -a_1; -a_4', -a_3 - a_2, -a_1'$) with $-a_4' \le -a_4, \le -a_3 \le -a_2 \le -a_1 \le a_1'$

$\tilde{A}$ (+) $(-\tilde{A})$= ($a_1, a_2, a_3, a_4; a_1' a_2, a_3\ a_4'$) + ($-a_4 - a_3, -a_2, -a_1; -a_4', -a_3 -$ $a_2, -a_1'$)

= ($a_1 - a_4, a_2 - a_3, a_3 - a_2, a_4 - a_1;\ a'_1 - a'_4, a_2 - a_3, a_3 - a_2, a'_4 - a'_1$)

$\neq 0$

Therefore we cannot find an additive inverse of $\tilde{A}$ .Here, property 4 failed.

Therefore the set of all Trapezoidal Intuitionistic Fuzzy number does not form a linear Space.

4. ARITHMETIC OPERATIONS OF TRAPEZOIDAL INTUITIONISTIC FUZZY NUMBER BASED ON (α, β)-CUT METHOD.

If $\tilde{A}$ is a trapezoidal intuitionistic fuzzy number and (α, β) – cut is given by $\tilde{A}_{\alpha,\beta} = \{x, \mu_{\tilde{A}}(x), v_{\tilde{A}}(x)) \mid x \in X, \mu_{\tilde{A}}(x) \ge \alpha, v_{\tilde{A}}(x) \le \beta\ \alpha, \beta \in [0,1]$

In other form, if $\tilde{A} = (a_1, a_2, a_3; a_1' a_2, a_3')$ is a Trapezoidal Intuitionistic Fuzzy number, then (α, β) level interval or (α, β) -cut is written as

$\tilde{A}_{\alpha,\beta} = \{[a_1 + \alpha(a_2 - a_1), a_4 - (a_4 - a_3)\alpha]; [a'_1 - \beta(a_2 - a'_1), a'_4 + (a'_4 - a_3)]\}, \alpha, \beta \in [0,1]$ where $\alpha + \beta \le 1$.

4.1 Special Properties of Trapezoidal Intuitionistic Fuzzy Number (TrIFN)

In this sectio, we discussed the properties of TrIFN.

4.1.1 Property

(a) If Trapezoidal Intuitionistic Fuzzy Number $\tilde{A}_{TrIFN}$= ($a_1, a_2, a_3, a_4; a_1' a_2, a_3, a_4'$)

and x = ka here k> 0 then$\tilde{x} = k\tilde{A}$ is TrIFN $\tilde{A}_{TrIFN}$= ($ka_1, ka_2, ka_3, ka_4; ka_1', ka_2, ka_3, ka_4'$).

(b)If $x = ka$ with $k < 0$ then $\tilde{x} = k\tilde{A}$ is a TrIFN $\tilde{A}_{TrIFN}$= ($ka_4, ka_3, ka_2, ka_1; ka_4', ka_3, ka_2, ka_1'$).

Proof: Let k> 0and the linear transformation x=ka, we can find the membership function and non membership function of TrIFN $\tilde{x} = k\tilde{a}$ by (α, β) − cut method.

α −cut of $\tilde{A}$,A_α=$[a_1 + \alpha(a_2 - a_1), a_4 - (a_4 - a_3)\alpha]$

$\mu_{\tilde{A}}(x) \geq \alpha \Rightarrow [a_1 + \alpha(a_2 - a_1), a_4 - (a_4 - a_3)\alpha]$ for any $\alpha \in [0,1]$

i.e $x \in [a_1 + \alpha(a_2 - a_1), a_4 - \alpha(a_4 - a_3)]$

so x= ka = $[ka_1 + \alpha(ka_2 - ka_1), ka_4 - \alpha(ka_4 - ka_3)]$

Thus the membership function of $\tilde{x} = k\tilde{A}$ is given by

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{x - k a_1}{ka_2 - ka_1} & for\ ka_1 \leq x \leq ka_2 \\ 1 & for\ ka_2 \leq x \leq ka_3 \\ \frac{ka_4 - x}{ka_4 - ka_3} & for\ ka_3 \leq x \leq ka_4 \\ 0 & otherwise \end{cases} \text{ and}$$

Hence the rule is proved for membership function.

For non membership function of, β- cut of $\widetilde{A}$ is

$v_{\tilde{A}}(x) \leq \beta$,$A_\beta$=$[a'_1 - \beta(a_2 - a'_1), a'_4 + (a'_4 - a_3)\beta]$ for any $\beta \in [0,1]$

$i.e\ x \in [a'_1 - \beta(a_2 - a'_1), a'_4 + (a'_4 - a_3)\beta]$

so x=ka= $[k'a_1 - \beta(ka_2 - ka'_1), ka'_4 + (ka'_4 - ka_3)\beta,]$

Thus the membership function of $\tilde{x} = k\tilde{A}$ is given by

$$v_{\tilde{A}}(x) = \begin{cases} \frac{ka'_1 - x}{ka_2 - ka'_1} & for\ ka'_1 \leq x \leq k a_2 \\ 0 & for\ ka_2 \leq x \leq ka_3 \\ \frac{x - k a'_4}{ka'_4 - ka_3} & for k a_3 \leq x \leq ka'_4 \\ 1 & otherwise \end{cases}$$

Hence the rule is proved for non-membership function.

Thus $\tilde{x} = k\tilde{A} = (ka_1, ka_2, ka_3, ka_4; ka'_1, ka_2, ka_3, ka'_4)$ is a TIFN when k> 0

(b) Similarly we can prove that $\tilde{x} = k\tilde{A}$ if $x = kx, k < 0$.

Here we get the membership and non-membership function as follows

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{x - k a_4}{ka_3 - ka_4} & for\ ka_4 \leq x \leq ka_3 \\ 1 & for\ ka_3 \leq x \leq ka_2 \\ \frac{ka_1 - x}{ka_1 - ka_2} & for\ ka_2 \leq x \leq ka_1 \\ 0 & otherwise \end{cases}$$

and

$$v_{\tilde{A}}(x) = \begin{cases} \frac{ka'_4 - x}{ka_3 - ka'_4} & for\ ka'_4 \leq x \leq k a_3 \\ 0 & for\ ka_3 \leq x \leq ka_2 \\ \frac{x - ka'_1}{ka'_1 - ka_2} & for k a_2 \leq x \leq ka'_1 \\ 1 & otherwise \end{cases}$$

Thus $\tilde{x} = k\tilde{A} = (ka_4, ka_3, ka_2, ka_1; ka'_4, ka_3, ka_2, ka'_1)$ is a TIFN when k< 0

4.1.2 Property

If $\tilde{A}$ is a TrIFN , where k and m are real numbers and the linear transformation $x = (k + m)a$, then $\tilde{x} = (k + m)\tilde{A}$ is also a TrIFN.

Proof:

If k and m are positive real numbers then k+m is a positive real number, k> 0 (proof is same as property 4.1(a))

If k and m are negative real numbers then k + m is a negative real number, k< 0 (proof is same as property 4.1(b))

If k and m are opposite in sign of real numbers then k + m is either positive or negative real number. k+m are positive then the property 4.1(a) or property (b) can be applied.

4.1.3 Property

If $\tilde{A} = (a_1, a_2, a_3, a_4; a'_1 a_2, a_3, a'_4)$ and $\tilde{B} = (b_1, b_2, b_3, b_4; b'_1 b_2, b_3, b'_4)$ are two TrIFN, their sum of $\widetilde{A}$ (+) $\widetilde{B}$ is also TrIFN $\widetilde{A}$ (+) $\widetilde{B}$= $((a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4; a'_1 + b'_1, a_2 + b_2, a_3 + b_3, a'_4 + b'_4)$

Proof:

With the transformation z = x +y, the membership (acceptance) function and non-membership (rejection) function of IFS $\tilde{Z} = \tilde{A}(+)\tilde{B}$ can be found by (α, β) –cut method.

α −cut for membership function of $\tilde{A}$,A_α=$[a_1 + \alpha(a_2 - a_1), a_4 - (a_4 - a_3)\alpha]$

$\mu_{\tilde{A}}(x) \geq \alpha \Rightarrow [a_1 + \alpha(a_2 - a_1), a_4 - (a_4 - a_3)\alpha]$ for any $\alpha \in [0,1]$

i.e $x \in [a_1 + \alpha(a_2 - a_1), a_4 - (a_4 - a_3)\alpha]$ and

α −cut for membership function of $\tilde{B}$,B_α=$[b_1 + \alpha(b_2 - b_1), b_4 - (b_4 - b)\alpha]$

$\mu_{\tilde{B}}(x) \geq \alpha \Rightarrow [b_1 + \alpha(b_2 - b_1), b_4 - (b_4 - b)\alpha]$ for any $\alpha \in [0,1]$

i.e $y \in [b_1 + \alpha(b_2 - b_1), b_4 - (b_4 - b)\alpha]$

so Z =X+Y $\in \left[a_1 + b_1 + \alpha\big((a_2 - a_1) + (b_2 - b_1)\big), a_4 + b_4 - (a_4 - a_3 + b_4 - b_3)\right]$

i.e α- cut of $\tilde{A} + \tilde{B} = \left[a_1 + b_1 + \alpha\big((a_2 - a_1) + (b_2 - b_1)\big), a_4 + b_4 - ((a_4 - a_3) + (b_4 - b_3))\right]$

So the membership function of acceptance (membership) of $\tilde{Z} = \tilde{A}(+)\tilde{B}$ is given by

$$\mu_{\tilde{A}(+)\tilde{B}}(Z) = \begin{cases} \frac{z - (a_1 + b_1)}{a_2 - a_1 + b_2 - b_1} & for\ a_1 + b_1 \leq x \leq a_2 + b_2 \\ 1 & for\ a_2 + b_2 \leq x \leq a_3 + b_3 \\ \frac{a_4 + b_4 - z}{a_4 - a_3 + b_4 - b_3} & for\ a_3 + b_3 \leq x \leq a_4 + b_4 \\ 0 & otherwise \end{cases}$$

Hence $\tilde{A}$ (+)$\tilde{B}$ is a TrIFN for membership function.

For non membership function of ,β- cut of $\widetilde{A}$ is

$v_{\tilde{A}}(x) \leq \beta$,$A_\beta$=$[a'_1 - \beta(a_2 - a'_1), a'_4 + \beta(a'_4 - a_3)]$ for any $\beta \in [0,1]$

$i.e\ x \in [a'_1 - \beta(a_2 - a'_1),, a'_4 + \beta(a'_4 - a_3)]$

α –cut for non membership function of $\tilde{B}$,B_α=$[b'_1 - \beta(b_2 - b'_1)\,, b'_4 + (\,b'_4 - b_3\,)\beta]$ $\forall\ \beta\ \in [0,1]$

$\mu_{\tilde{B}}(x) \geq \beta \Rightarrow [b'_1 - \beta(b_2 - b'_1)\,, b'_4 + (\,b'_4 - b_3\,)\beta]$ for any $\beta \in [0,1]$

i.e $y \in [b'_1 - \beta(b_2 - b'_1), b'_4 + (\,b'_4 - b_3\,)\beta]$

so $Z=X+Y \in \left[a'_1 + b'_1 + \beta\big((a_2 - a'_1) + (b_2 - b'_1)\big), a'_4 + b'_4 + \beta((\,a'_4 - a_3\,) + (\,b'_4 - b_3\,))\right]$

i.e α- cut of $\tilde{A} + \tilde{B} = \left[a'_1 + b'_1 + \beta\big((a_2 - a'_1) + (b_2 - b'_1)\big), a'_4 + b'_4 + \beta((\,a'_4 - a_3\,) + (\,b'_4 - b_3\,))\right]$

So the non membership function of rejection (membership) of $\tilde{Z} = \tilde{A}(+)\tilde{B}$ is given by

$$v_{\tilde{A}(+)\tilde{B}}(Z) = \begin{cases} \frac{a'_1 + b'_1 - z}{a_2 - a'_1 + b_2 - b'_1} & for\, a'_1 + b'_1 \leq x \leq a_2 + b_2 \\ 0 & for\ a_2 + b_2 \leq x \leq a_3 + b_3 \\ \frac{z - (a'_4 + b'_4)}{a'_4 - a_3 + b'_4 - b_3} & for\ a_3 + b_3 \leq x \leq a'_4 + b'_4 \\ 1 & otherwise \end{cases},$$

Hence the addition rule is proved for non-membership function,

Thus $\tilde{A}\ (+)\ \tilde{B}$= $(\,(\,a_1 + b_1, a_2 + b_2, a_3 + b_3\,, a_4 + b_4\,;\ a'_1 + b'_1, a_2 + b_2, a_3 + b_3\,, a'_4 + b'_4)$ is a TrIFN.

4.1.4 Property

If $\tilde{A} = (a_1, a_2, a_3\,, a_4; a'_1 a_2, a_3\,, a'_4)$ and $\tilde{B} = (b_1, b_2, b_3, b_4; b'_1 b_2, b_3, b'_4)$ are two TrIFN, their product $\tilde{A}(\times)\tilde{B}$ also TrIFN $\tilde{A}(\times)\tilde{B}$ =$(\,a_1 b_1, a_2 b_2, a_3 b_3\,, a_4 b_4; a_1{}' b_1{}', a_2 b_2, a_3 b_3 a_4{}' b_4{}')$

Proof: With the transformation z = x × y, the membership (acceptance) function and non-membership (rejection) function of IFS $\tilde{Z} = \tilde{A}(\times)\tilde{B}$ can be found by (α, β) –cut method.

α –cut for membership function of $\tilde{A}$,A_α=$[a_1 + \alpha(a_2 - a_1), a_4 - (a_4 - a_3\,)\alpha]$

$\mu_{\tilde{A}}(x) \geq \alpha \Rightarrow [a_1 + \alpha(a_2 - a_1), a_4 - (a_4 - a_3\,)\alpha]$ for any $\alpha \in [0,1]$

i.e $x \in [a_1 + \alpha(a_2 - a_1), a_4 - (a_4 - a_3\,)\alpha]$ and

α –cut for membership function of $\tilde{B}$,B_α = $[b_1 + \alpha(b_2 - b_1), b_4 - (b_4 - b_3\,)\alpha]$

$\mu_{\tilde{B}}(x) \geq \alpha \Rightarrow [b_1 + \alpha(b_2 - b_1), b_4 - (b_4 - b_3)\alpha]$ for any $\alpha \in [0,1]$

i.e $y \in [b_1 + \alpha(b_2 - b_1), b_4 - (b_4 - b_3\,)\alpha]$

so $Z = X\ (\times)\ Y \in [(\,a_1 + \alpha(a_2 - a_1))\ (b_1 + \alpha(b_2 - b_1)), (a_4 - \alpha(a_4 - a_3\,)(b_4 - (b_4 - b_3\,)\alpha]$

$\mu_{\tilde{A}(\times)\tilde{B}}, v_{\tilde{A}(\times)\tilde{B}}$

So the membership function of acceptance (membership) of $\tilde{Z} = \tilde{A}(\times)\tilde{B}$ is given by

$$\mu_{\tilde{A}(\times)\tilde{B}}(Z) = \begin{cases} \frac{-B_1 + \sqrt{{B_1}^2 - 4A_1(\,a_1 b_1 - z\,)}}{2A_1} & for\ a_1 b_1 \leq z \leq a_2 b_2 \\ 1 & for\ a_2 b_2 \leq z \leq a_3 b_3 \\ \frac{B_2 - \sqrt{{B_2}^2 - 4A_2(a_4 b_4 - z)}}{2A_2} & for\ a_3 b_3 \leq z \leq a_4 b_4 \\ 0 & otherwise \end{cases}$$

Where $A_1 = (a_2 - a_1)(b_2 - b_1)$

$$B_1 = b_1(a_2 - a_1) + a_1(b_2 - b_1)$$

$$A_2 = (a_4 - a_3)(b_4 - b_3)$$

And $B_2 = -(b_4(a_4 - a_3) + a_4(b_4 - b_3))$

For non membership function of,β- cut of $\widetilde{A}$ is

$v_{\tilde{A}}\ (x) \leq \beta, A_\beta$=$[a'_1 - \beta(a_2 - a'_1), a'_4 + \beta(a'_4 - a_3)]$ for any $\beta \in [0,1]$

$i.e\ x \in [a'_1 - \beta(a_2 - a'_1), a'_4 + \beta(a'_4 - a_3)]$

α –cut for non membership function of $\tilde{B}$,B_α=$[b'_1 - \beta(b_2 - b'_1), b'_4 + \beta(b'_4 - b_3)]$

$\mu_{\tilde{B}}(x) \geq \beta \Rightarrow [b'_1 - \beta(b_2 - b'_1), b'_4 + \beta(b'_4 - b_3)]$ for any $\beta \in [0,1]$

i.e. $y \in [b'_1 - \beta(b_2 - b'_1), b'_4 + \beta(b'_4 - b_3)]$

So $Z = X\ (\times)\ Y \in [(a'_1 - \beta(a_2 - a'_1))\big(\,b'_1 - \beta(b_2 - b'_1)\big), (a'_4 + \beta(a'_4 - a_3))(\,b'_4 + \beta(b'_4 - b_3))]$

So the non-membership rejection function of $\tilde{Z} = \tilde{A}(\times)\tilde{B}$ is given by

$$v_{\tilde{A}(\times)\tilde{B}}(Z) = \begin{cases} \frac{B'_1 - \sqrt{{B'_1}^2 - 4A'_1(\,a'_1 b'_1 - z\,)}}{2A'_1} & for\ a'_1 b'_1 \leq z \leq a_2 b_2 \\ 0 & for\ a_2 b_2 \leq z \leq a_3 b_3 \\ \frac{-B'_2 + \sqrt{{B'_2}^2 - 4A'_2(a'_4 b'_4 - z)}}{2A'_2} & for\ a_3 b_3 \leq z \leq a'_4 b'_4 \\ 1 & otherwise \end{cases}$$

Where $A'_1 = (a_2 - a'_1)(b_2 - b'_1)$

$$B'_1 = b_2(a_2 - a'_1) + a_2(b_2 - b'_1)$$

$$A'_2 = (a'_4 - a_3)(b'_4 - b_3)$$

And $B'_2 = (b_3(a'_4 - a_3) + a_3(b'_4 - b_3)$

Hence the multiplication rule is proved for non-membership function,

Thus $\widetilde{A}\ (\times)\ \widetilde{B}$ = $(\,a_1 b_1, a_2 b_2, a_3 b_3, a_4 b_4; a_1{}' b_1{}', a_2 b_2, a_3 b_3\,, a'_4 b'_4))$ is a TrIFN. As shown in Fig. 3 with dotted line.

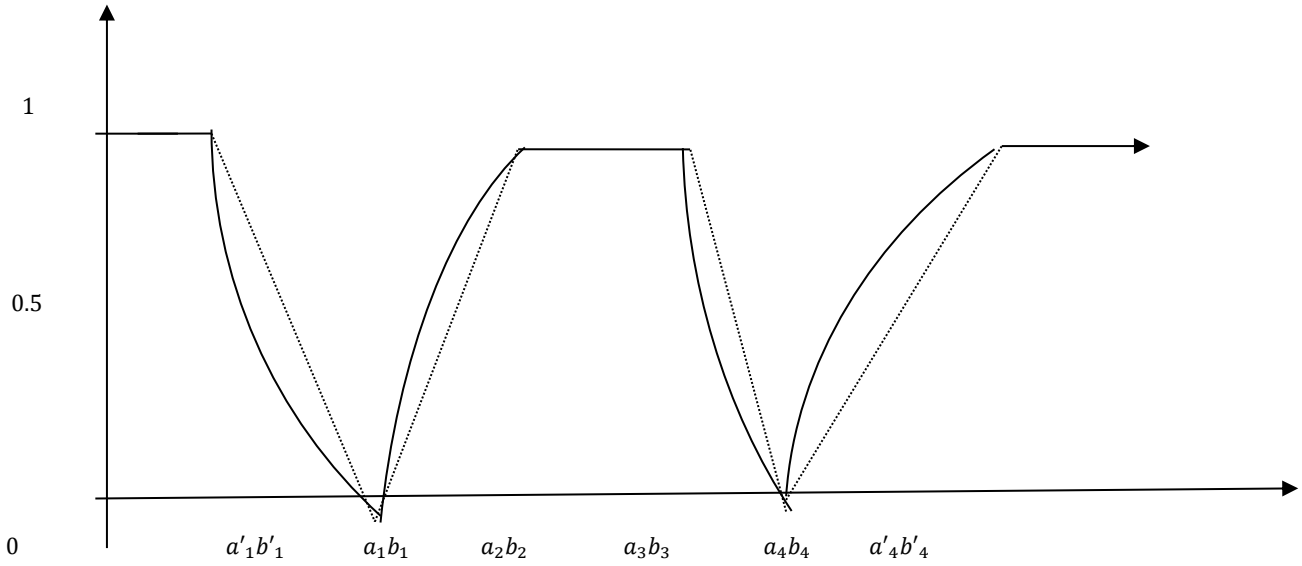

Fig.3 Membership and non-membership function for product of two TrIFNs

5. NUMERICAL EVALUATION AND AN ELECTRIC NETWORK MODEL PRESENTED

1.Let $\tilde{A} = (1,2,3,4; 0.5\ 2,3,4.5)$ and $\tilde{B} =(2,3,4,5; 1,3,4,5.5)$ are two TrIFN then the sum $\tilde{A}(+)\tilde{B}$ is defined by $\tilde{A}(+)\tilde{B}$ = (3,5,7,9;1.5,5,7,10) with membership and non-membership function are as follows:

$$\mu_{\tilde{A}(+)\tilde{B}}(x) = \begin{cases} \frac{x-3}{2} & for\ 3 \le x \le 5 \\ 1 & for\ 5 \le x \le 7 \\ \frac{9-x}{2} & for\ 7 \le x \le 9 \\ 0 & otherwise \end{cases} \quad \text{and}$$

For non-membership function,

$$v_{\tilde{A}(+)\tilde{B}}(x) = \begin{cases} \frac{1.5-x}{3.5} & for\ 1.5 \le x \le 5 \\ 0 & for\ 5 \le x \le 7 \\ \frac{x-10}{3} & for\ 7 \le x \le 10 \\ 1 & otherwise \end{cases}$$

2.Let $\tilde{A} = (1,2,3,4; 0.5\ 2,3,\ 4.5)$ and $\tilde{B} = (2,3,4\ ,5; 1,3,4,5.5)$ are two TrIFN then their product $\tilde{A}(\times)\tilde{B}$ is defined by $\tilde{A}(\times)\tilde{B} = (2,6,12,20\ ;0.5,6,12,24.75)$ with membership and non-membership function are as follows:

$$\mu_{\tilde{A}(\times)\tilde{B}}(x) = \begin{cases} \frac{-3+\sqrt{9-4(2-x)}}{2} & \text{for } 2 \le x \le 6 \\ 1 & \text{for } 6 \le z \le 12 \\ \frac{-9-\sqrt{81-4(20-z)}}{2} & \text{for } 12 \le z \le 20 \\ 0 & \text{otherwise} \end{cases} \quad \text{and}$$

$$v_{\tilde{A}(\times)\tilde{B}}(x) = \begin{cases} \frac{8.5-\sqrt{72.25-12(0.5-x)}}{6} & \text{for } 0.5 \le x \le 6 \\ 0 & \text{for } 6 \le z \le 12 \\ \frac{-10.5+\sqrt{110.25-9(24.75-x)}}{4.5} & \text{for } 12 \le z \le 24.75 \\ 1 & \text{otherwise} \end{cases}$$

Above $\tilde{A}(\times)\tilde{B}$ is a trapezoidal intuitionistic fuzzy number .It can be approximated to TrIFN as $\tilde{A}(\times)\tilde{B}$ = (2, 6,12, 20;0.5,6,12,24.75) with membership and non-membership function are as follows

$$\mu_{\tilde{A}(x)\tilde{B}}(x) = \begin{cases} \frac{x-2}{4} & for\ 2 \le x \le 6 \\ 1 & for\ 6 \le x \le 12 \\ \frac{20-x}{8} & for\ 12 \le x \le 20 \\ 0 & otherwise \end{cases} \quad \text{and}$$

For non-membership function,

$$v_{\tilde{A}(\times)\tilde{B}}(x) = \begin{cases} \frac{0.5-x}{5.5} & for\ 0.5 \le x \le 6 \\ 0 & for\ 6 \le x \le 12 \\ \frac{x-24.75}{12.75} & for\ 12 \le x \le 24.75 \\ 1 & otherwise \end{cases}$$

For application, an electric network model presented by Dhillon [8] of a dark room is taken. The system models a dark, windowless room containing one master switch and four light bulbs, to develop a desired event in an illuminated dark room. Thus, in this case, the room can only be brightened if there is availability of electricity. If the switch is Ok, the four light bulbs are not burnt out. By using this application is developed. Each event is considered as TrIFN.

R_1 represent no fuse failure of electric supply of dark room,

R_2 represent no power failure of electric supply of dark room,

R_3 represent the electric switch of dark room,

R_4 represent the bulb no. 1 of dark room,

R_5 represent the bulb no. 2 of dark room,

R_6 represent the bulb no. 3 of the dark room,

R_7 represent the bulb no. 4 of the dark room,

R_8 represent the electric supply of dark room,

R_9 represent the bulbs not burnt.

R_{10} represent the desired dark room,

All imprecise components R_j are represented by TrIFN $(a_{j1}, a_{j2}, a_{j3}\ a_{j4}\ ;\ a'_{j1}, a_{j2}, a_{j3}\ , a'_{j4})$ for j= 1, 2… 7. Let us calculate the desired event i.e dark room with illumination. The value of the event electricity supply,

$R_8 = R_1(\times)R_2 =$
$($
$a_{11}a_{21}, a_{12}a_{22}, a_{13}a_{23}, a_{14}a_{24}; a'_{11}a'_{21}, a_{12}a_{22}, a_{13}a_{23}, a'_{14}a'_{24}$
$)$ ... (5.1)

It is an approximated TIFN.

Similarly, the value of event four bulbs not burnt,

$R_9 = 1(-)\ \{(1(-)R_4)(1(-)R_5)(1 - R_6)(1 - R_7\}$.It is an approximated TrIFN as follows

$\cong \{1 - \prod_{j=4}^{7} 1 - a_{j1}), 1 - \prod_{j=4}^{7} 1 - a_{j2}), 1 - \prod_{j=4}^{7} 1 - a_{j3}), 1 - \prod_{j=4}^{7}(1 - a_{j4})\ ;\ 1 - \prod_{j=4}^{7} 1 - a'_{j1}), 1 - \prod_{j=4}^{7} 1 - a_{j2}) 1 - \prod_{j=4}^{7} 1 - a_{j3}), 1 - \prod_{j=4}^{7}(1 - a'_{j=4}\}$ ... (5.2)

By substituting the above two calculated values and the given data R_3, we get the value for the, dark room,

$R_{10} = R_1(\times)R_2(\times)R_3(\times)[1(-)\{(1(-)R_4)(1(-)R_5)(1 - R_6(1 - R_7\}]$.

It is an approximated TIFN as follows

$\cong \{a_{11}a_{21}a_{31}, a_{41}(1 - \prod_{j=4}^{7} 1 - a_{j1}), a_{12}a_{22}a_{32}a_{42}(1 - \prod_{j=4}^{7} 1 - a_{j2}), a_{13}a_{23}a_{33}a_{43}(1 - \prod_{j=4}^{7} 1 - a_{j3}), a_{14}a_{24}a_{34}a_{44}(1 - \prod_{j=4}^{7}(1 - a_{j4}));\ a'_{11}a'_{21}a'_{31}a'_{14}(1 - \prod_{j=4}^{7} 1 - a'_{j1}), a_{12}a_{22}a_{32}a_{42}(1 - \prod_{j=4}^{7} 1 - a_{j2}), a_{13}a_{23}a_{33}a_{43}(1 - \prod_{j=4}^{7} 1 - a_{j3})\ , a'_{14}a'_{24}a'_{34}a'_{44}(1 - \prod_{j=4}^{7}(1 - a_{j4})\}$ ..., (5, 3)

Let the component of events are

R_1= (0.65, 0.70, 0.75, 0.80; 0.55, 0.70, 0.75, 0.85),

R_2= (0.70, 0.75, 0.85, 0.90; 0.65, 0.75, 0.85, 0.95),

R_3= (0.82, 0.85, 0.88, 0.92; 0.80, 0.85, 0.88 0.96),

R_4= (0.85, 0.86, 0.87, 0.89; 0.83, 0.86, 0.87, 0.91)

R_5= (0.65, 0.75, 0.85, 0.90; 0.60, 0.75, 0.85, 0.95)

R_6= (0.89, 0.92, 0.94, 0.96; 0.86, 0.92, 0.94, 0.97),

R_7= (0.80, 0.82, 0.84, 0.86; 0.75, 0.82, 0.84, 0.88)

So results of equation (5.1) and (5.2) are as follows,

R_8= (0.455, 0.525, 0.6375, 0.72; 0.3575, 0.525, 0.6375, 0.8075) and

R_9= (0.998845, 0.999496, 0.9998128, 0.9999384; 0.99762, 0.999496, 0.9998128, 0.9999784),

By substituting the above R_8 and R_9 the given data value is (5.3), the value of desired dark room,

R_{10} = (0.37266907, 0.44602509, 0.560894981, 0.662359196; 0.285311932, 0.44602509, 0.560894981, 0.775183411). As shown in Fig. 4 with dotted line

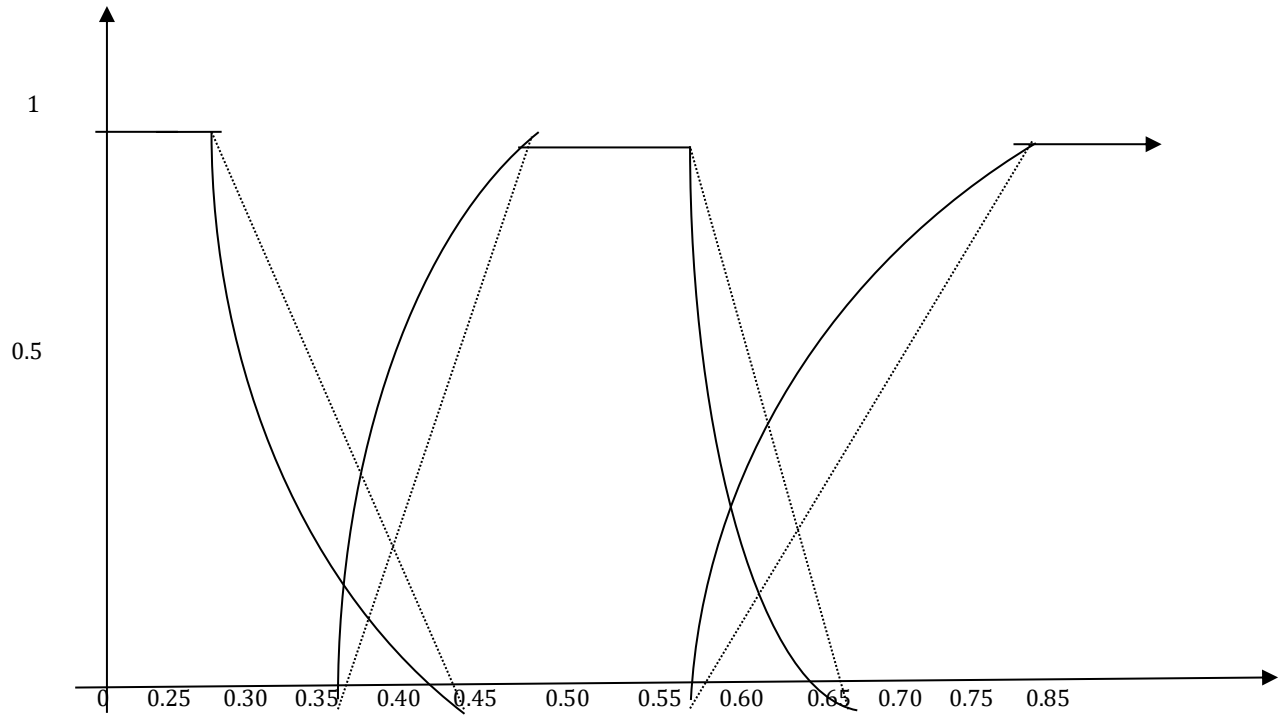

Fig.4 desired event in an illuminated dark room.

6. CONCLUSION

In this paper, definition of positive trapezoidal intuitionistic fuzzy number and negative trapezoidal intuitionistic fuzzy number are proposed.Also arithmetic operation proposed TrIFN is evaluated based on intuitionistic fuzzy (α, β) -cut method. The set of all trapezoidal fuzzy numbers form a linear space,but the set of all trapezoidal intuitionistic fuzzy numbers does not form a linear space are proved. Membership and non-membership function are discussed using the (α, β) –cut method of TrIFN. Later we will try to calculate the reliability evaluation using trapezoidal intuitionistic fuzzy numbers with the help of trapezoidal fuzzy linear space.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

Dedication

This work is lovingly dedicated to my little son Ivyavan Chamuah, whose innocent smile unconditional love, joyful presence have been a constant source of inspiration throughout the work.

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