A Deterministic Inventory Model for Deteriorating Items with Fractional Time-Dependent Demand, Time-Dependent Deterioration Rate and Backlogged Shortages

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ABSTRACT
This paper develops a deterministic single-item inventory model for deteriorating items with fractional time-dependent demand and a time-dependent deterioration rate. Demand is represented by $D(t) = \alpha + \beta t^{(m/n)}$, while deterioration is represented by $\lambda(t) = \gamma t^{(m/n-1)}$. Shortages are permitted and completely backlogged; lead time is zero, replenishment is instantaneous, and the holding cost is constant. The inventory cycle is divided into a positive-inventory phase and a shortage phase. The governing differential equations are solved using boundary conditions, and a total-cost-per-unit-time function is formulated from ordering, purchasing, holding, deterioration, and shortage costs. A numerical optimization and parameter-sensitivity study are used to evaluate the policy. Under the baseline parameter set, the optimized policy is $\Theta * = 1.2114$ and $T * = 2.8253$, with minimum average total inventory cost $TIC * = 123.1397$. Additional parameter scenarios are included to examine how demand and cost assumptions affect the optimized policy.

Keywords
Deteriorating inventory; fractional time-dependent demand; time-dependent deterioration; fully backlogged shortages; inventory cost; numerical optimization; sensitivity analysis.

1. INTRODUCTION
Inventory management is a primary concern in operations research. One of the most difficult types of inventory items to deal with is deteriorating items. Deteriorating items are inventory items that suffer from spoilage, decay, evaporation and obsolescence which reduce the quality, utility or quantity of the item over time. Typical examples include fresh food, fruits, vegetables, pharmaceutical drugs, blood banks, chemicals, and electronic components, which lose value during storage. Deterioration directly affects replenishment decisions, cost structures, and service levels; thus it is necessary to explicitly incorporate it into any realistic inventory model [1, 2].

2. LITERATURE REVIEW
Deteriorating inventory systems have received considerable attention in inventory theory because many products experience a loss of quantity, quality, utility, or economic value during storage. Such deterioration may arise from spoilage, decay, evaporation, obsolescence, or other physical and functional degradation. Consequently, deterioration has an important influence on replenishment decisions, inventory levels, and total inventory cost.

Ghare and Schrader [1] developed one of the early inventory models for deteriorating items by considering an exponentially decaying inventory system. Their work established an important foundation for subsequent research on inventory systems in which inventory depletion occurs through both demand and deterioration. Covert and Philip [2] subsequently introduced a two-parameter Weibull distribution to describe deterioration, thereby providing a more flexible representation of the deterioration process. Shah and Jaiswal [3] considered an order-level inventory model with a constant rate of deterioration. These early studies established the importance of explicitly incorporating deterioration into inventory models.

2.1 Demand Patterns in Inventory Models
The demand rate is one of the principal components determining the behaviour of an inventory system. Classical inventory models generally assume a constant demand rate; however, this assumption may not adequately represent products for which demand changes during the replenishment cycle. Silver and Meal [4] investigated inventory systems with varying demand rates and demonstrated the importance of incorporating time variation into inventory decisions.

Subsequent studies considered several forms of time-dependent demand. Dave and Patel [5] developed a deteriorating-inventory model with time-proportional demand, while Sachan [6] extended the analysis to an inventory system with time-proportional demand and shortages. Mandal and Phaujdar [7] considered an inventory model in which the consumption rate depends on the inventory level. Baker and Urban [8] similarly investigated an inventory system with inventory-level-dependent demand. These studies demonstrated that demand need not be independent of time or inventory level.

Demand has also been modeled as a function of selling price and other economic factors. Wee [9] considered a replenishment policy for items with price-dependent demand and a varying deterioration rate. Cheng and Wang [10] investigated deteriorating items with a trapezoidal demand rate. Manna and Chaudhuri [11] developed an EOQ model with ramp-type demand, a time-dependent deterioration rate, unit production cost, and shortages. These studies indicate that the selection of an appropriate demand function can substantially affect the resulting inventory policy.

More recently, fractional or non-integer power functions have been considered for representing nonlinear time-dependent demand. Kusum and Vinod Kumar [22] developed an inventory model with fractionally time-dependent demand and variable holding cost. A fractional demand formulation

provides additional flexibility compared with simple linear demand because the fractional exponent can represent nonlinear changes in demand over time.

2.2 Deterioration Rate in Inventory Models

The deterioration rate is another important component of deteriorating-inventory models. Early models commonly assumed a constant deterioration rate because of its mathematical simplicity. Ghare and Schrader [1] considered exponential decay, while Shah and Jaiswal [3] investigated an inventory system with a constant deterioration rate. Covert and Philip [2] and Philip [12] introduced Weibull-type deterioration, providing greater flexibility in representing deterioration behaviour.

The literature subsequently progressed toward deterioration rates that vary with time. Wee [9] considered a varying rate of deterioration together with price-dependent demand. Manna and Chaudhuri [11] developed a model in which the deterioration rate is time dependent and the demand rate follows a ramp-type pattern. Chang and Dye [13] studied deteriorating items with time-varying demand and partial backlogging. These studies demonstrate that assuming a constant deterioration rate is not the only possible formulation and that deterioration may change during the storage period.

Time-dependent deterioration has therefore become an established modeling approach in deteriorating-inventory research. The present study builds on this line of research by considering a specific non-constant deterioration function within a fractional-demand inventory framework. The deterioration rate in the proposed model is represented by

$$\lambda(t) = \gamma\, t^{\frac{m}{n}-1}$$

Where $\gamma > 0$ is the deterioration parameter and m/n is the fractional exponent associated with the demand function. Thus, deterioration is allowed to vary with time rather than remaining constant throughout the inventory cycle.

2.3 Shortages and Backlogging

Shortages are another important consideration in inventory management. When inventory becomes zero before the next replenishment, customer demand may either be lost or accumulated for subsequent fulfillment. The latter situation is generally referred to as backlogging.

Dave and Patel [5] considered shortages in a deteriorating-inventory system with time-proportional demand, while Sachan [6] incorporated complete backlogging into a related inventory framework. Chang and Dye [13] considered partial backlogging in an inventory model for deteriorating items with time-varying demand. These studies demonstrate the importance of explicitly modeling customer behaviour during shortage periods.

The present study assumes fully backlogged shortages. Under this assumption, demand occurring during the shortage period is not lost but is fulfilled after replenishment. This permits the inventory cycle to be divided into a positive-inventory period and a shortage period, with the corresponding holding, deterioration, and shortage costs incorporated into the total inventory cost.

2.4 Economic and Operational Extensions

Several researchers have incorporated additional economic and operational factors into deteriorating-inventory models. Kumar and Kumar [14] considered deteriorating items under inflation and permissible delay in payments with partial backlogging. Teng and Chang [15] investigated economic production quantity models for deteriorating items with price- and stock-dependent demand. Goyal [16] developed an economic order quantity model under permissible delay in payments. Aggarwal and Jaggi [17] extended the analysis to deteriorating items under permissible delay in payments. Hwang and Shinn [18] investigated pricing and lot-sizing decisions for exponentially deteriorating products under permissible delay in payment conditions. Chung and Huang [19] further examined optimal cycle time in an economic production quantity model with permissible delay in payments. Inflation is another important economic factor in inventory decisions. Buzacott [20] investigated economic order quantities under inflation, while Misra [21] considered optimal inventory management under inflation. These studies established the importance of considering the time value of money when inventory decisions are made over extended planning periods. Rajan and Uthayakumar [23] subsequently developed a pricing and replenishment model for instantaneous deteriorating items with backlogging, trade credit, and inflation. Such extensions demonstrate the wide range of economic and operational characteristics that can be incorporated into deteriorating-inventory models.

Although these studies improve the practical applicability of inventory models, they primarily focus on additional economic factors such as inflation, trade credit, pricing, and permissible delay in payment. The present study instead focuses on the mathematical representation of a time-varying deterioration mechanism within a fractional-demand and shortage framework.

2.5 Fractional Time-Dependent Demand

Fractional time-dependent demand provides an alternative to conventional integer-order polynomial demand functions. In such a formulation, the demand rate may be expressed as

$$D(t) = \alpha + \beta\, t^{(m/n)},$$

where α represents the initial demand component, β is the time-demand coefficient, and m/n is a fractional exponent.

Kusum and Vinod Kumar [22] investigated an inventory model with fractionally time-dependent demand and variable holding cost. Their study demonstrates the applicability of fractional demand functions to inventory systems and provides a relevant foundation for the demand structure adopted in the present study. However, fractional demand alone is not claimed as the principal novelty of the present model. Instead, the fractional demand function is used as part of a broader formulation in which the deterioration rate is also allowed to vary with time.

2.6 Research Gap

The existing literature demonstrates that several important characteristics of deteriorating inventory systems have been studied independently or in different combinations. These include constant deterioration rates [1,3], Weibull-type deterioration [2,12], time-dependent deterioration [9,11], time-dependent and nonlinear demand [4–11], shortages and backlogging [5,6,13], fractional time-dependent demand [22], and additional economic considerations such as inflation and trade credit [14–21,23].

The literature also confirms that time-dependent deterioration is not itself a completely unexplored concept. Therefore, the contribution of the present study should not be stated as the first introduction of time-dependent deterioration into inventory theory. Instead, the research gap addressed in this study concerns the specific integration of a non-constant

deterioration rate with fractional time-dependent demand and fully backlogged shortages in the deterministic single-item inventory framework considered here.

In the proposed model, demand is represented by

$$D(t) = \alpha + \beta\, t^{(m/n)}, \quad \text{.....1}$$

while the deterioration rate is represented by

$$\lambda(t) = \gamma\, t^{(m/n\,-\,1)} \quad \text{.....2}$$

Consequently, both demand and deterioration vary with time, and the deterioration rate is directly linked to the fractional time structure used in the demand function. The model also permits shortages and assumes that all shortages are fully backlogged. The proposed formulation therefore extends the existing literature by bringing these features together within a single inventory model.

2.7 Contribution of the Present Study

Based on the above literature, the principal contribution of the present study is the analysis of a non-constant, time-dependent deterioration rate within a fractional time-dependent demand model with fully backlogged shortages.

The proposed deterioration function, $\lambda(t) = \gamma\, t^{(m/n\,-\,1)}$, allows the deterioration rate to change over the inventory cycle instead of assuming a fixed deterioration percentage. This provides a more flexible representation of inventory systems in which the rate of deterioration changes as storage time progresses. The resulting model is solved using the standard inventory differential equations and total-cost formulation. A numerical example and sensitivity analysis are subsequently used to examine the influence of the model parameters on the optimal inventory policy and total inventory cost.

3. OBJECTIVES OF THE STUDY

The objectives of this research are:

1. To present a deterministic inventory model with fractional time-dependent demand that generalizes classical integer-order demand models [22] and incorporates a time-dependent deterioration.
2. To formulate and solve first-order differential equations governing inventory dynamics in the positive inventory and shortage phases and construct a comprehensive total inventory cost (TIC) function integrating all cost components: ordering, purchasing, holding, deterioration, and shortage costs.
3. To determine the optimal replenishment policy via numerical optimization of the nonlinear TIC function and validate the model through a detailed numerical example and conduct sensitivity analysis on key parameters.

4. NOTATION AND ASSUMPTIONS

4.1 Notation

The notation used in the earlier model of Kusum and Vinod Kumar [22] is as follows: D for the demand rate, for the deterioration cost, A for the ordering cost, for the holding cost, Q for the order quantity, for the deterioration rate, for the purchase cost per unit, Ω for the time at which the inventory level reaches zero, T for the length of the cycle, for the shortage cost per unit time, for the total inventory cost, and the study(t) for the inventory level. The new notations introduced in this paper are given below.

Table 1. Notation and description of model variables.

Symbol	Description
$D(t)$	Demand rate at time t
α, β	Non-negative demand parameters ($\alpha \geq 0$, $\beta \geq 0$)
m, n	Natural numbers defining the fractional exponent m/n
$\lambda(t)$	Time-dependent deterioration rate: $\lambda(t) = \gamma t^{\left(\frac{m}{n}\right)-1}$.
γ	Deterioration coefficient, $0 < \gamma < 1$
$I(t)$	Inventory level at time t
Θ	Time at which inventory level reaches zero
T	Length of one complete replenishment cycle
Q	Order quantity per cycle
C_O	Fixed ordering cost per cycle
C_P	Purchasing cost per unit
C_H	Holding cost per unit per unit time (constant)
C_S	Shortage cost per unit per unit time
C_D	Deterioration cost per unit
TIC	Total inventory cost per unit time
TC	Total inventory cost per cycle

4.2 Assumptions

As in Kusum and Vinod Kumar [22], this model assumes that demand is a fractional power function of time, shortages are fully backlogged, lead time is zero, a single item is considered, and the ordering cost and shortage cost per unit are fixed. This paper modifies [22] by treating the holding cost as constant rather than linearly increasing with time, and by treating the deterioration rate as time-dependent rather than constant as assumed in [22], while retaining the fractional-demand structure. Assumptions of this paper are following,

1. The demand rate is a fractional power function of time: $D(t) = \alpha + \beta t^{m/n}$, where $\alpha \geq 0, \beta \geq 0$, and m and n are natural numbers [22].
2. The deterioration rate is time-dependent: $\lambda(t) = \gamma t^{\left(\frac{m}{n}\right)-1}$, where $0 < \gamma < 1, m/n > 0$.
3. The holding cost per unit per unit time is constant.
4. Shortages are allowed and are completely backlogged (no sales lost).
5. The lead time is assumed to be zero, and replenishment occurs instantaneously.
6. This model considers only a single inventory item.
7. The ordering cost C_O per replenishment cycle is also fixed.
8. The shortage cost C_S per unit per unit time is constant.
9. The time Θ at which the inventory reaches zero satisfies $0 \leq \Theta \leq T$.
10. Deteriorated items are neither repaired nor replaced during the inventory cycles.

5. MATHEMATICAL FORMULATION

The positive-inventory phase is governed by a first-order differential equation that accounts for demand and time-dependent deterioration. The shortage phase is governed by

demand alone because all demand after the zero-inventory time is backlogged. This two-phase formulation is used to derive the inventory trajectory and the cost function.

5.1 Phase 1: Positive Inventory Period

$\frac{d}{dt}I_1(t) + \gamma t^{(\frac{m}{n}-1)} I1(t) = -(\alpha + \beta t^{m/n}),\ 0 \le t \le \Theta$
(Positive Inventory Period) ……..(3)

This is a first-order linear ODE (3) of the form $dI/dt + P(t)I = Q(t)$, where

$$P(t) = \gamma t^{(\frac{m}{n}-1)},\ Q(t) = -(\alpha + \beta t^{m/n})$$

Step 1: Integrating Factor: IF $= e^{\int P(t)dt}$

$$\text{IF} = e^{\int \gamma t^{(\frac{m}{n}-1)}dt}$$

Now, $\int \gamma t^{(\frac{m}{n}-1)}dt = \gamma \frac{n}{m} t^{m/n}$

$$\text{IF} = e^{\gamma \frac{n}{m} t^{(\frac{m}{n})}} \quad \text{……(4)}$$

Step 2: Multiply both sides by IF from equation (4) and integrate it we get,

$$\frac{d\,[I_1(t) e^{\gamma \frac{n}{m} t^{(\frac{m}{n})}}]}{dt} = -(\alpha + \beta t^{\frac{m}{n}}) e^{\gamma \frac{n}{m} t^{(\frac{m}{n})}} \quad \text{…….. (5)}$$

Step 3: Integrate both sides of equation (5) (the integral is not in an elementary closed form; it is expressed in definite integral form) we get,

$$I_1(t) e^{\gamma \frac{n}{m} t^{(\frac{m}{n})}} = -\int \alpha + \beta t^{m/n}) e^{\gamma \frac{n}{m} t^{(\frac{m}{n})}} dt + C_1 \quad \text{…...(6)}$$

Step 4: Apply the boundary condition $I_1(\Theta) = 0$ in equation (6):

$$0 = -\int_0^{\Theta} (\alpha + \beta s^{\frac{m}{n}})\, e^{\gamma \frac{n}{m} s\wedge(\frac{m}{n})} ds + C_1$$

$$C_1 = \int_0^{\Theta} \left(\alpha + \beta s^{\frac{m}{n}}\right) e^{\gamma \frac{n}{m} s^{\frac{m}{n}}} ds$$

Final solution for Phase 1:

$$I_1(t) = e^{-\gamma \frac{n}{m} t^{(\frac{m}{n})}} \int_t^{\Theta} \left(\alpha + \beta s^{\frac{m}{n}}\right) e^{\gamma \frac{n}{m} s^{\frac{m}{n}}} ds,$$
…….(7)

$0 \le t \le \Theta$

At $t = 0$, we get the initial inventory level:

$$I_1(0) = \int_0^{\Theta} \left(\alpha + \beta s^{\frac{m}{n}}\right) e^{\gamma \frac{n}{m} s^{\frac{m}{n}}} ds = \text{E} \quad \text{…….(8)}$$

5.2 Phase 2: Shortage Period

$$\frac{dI_2(t)}{dt} = -(\alpha + \beta t^{\frac{m}{n}}),$$
……(9) $\quad \Theta \le t \le T$
(Shortage Period)

Integrate equation (9),

$$\int dI_2(t) = \int -\left(\alpha + \beta t^{\frac{m}{n}}\right) dt + C_2$$

$$I_2(t) = -\alpha t - \beta \left(\frac{n}{m+n}\right) t^{(\frac{m+n}{n})} + C_2 \quad \text{…….. (10)}$$

Integrating and applying the boundary condition $I_2(\Theta) = 0$ in equation (10):

$$C_2 = \alpha \Theta + \beta \left(\frac{n}{n+m}\right) \Theta^{\frac{(m+n)}{n}}$$

Final solution for phase 2,

$$I_2(t) = \alpha(\Theta - t) + \beta \left(\frac{n}{m+n}\right) (\Theta^{\frac{m+n}{n}} - t^{\frac{m+n}{n}})$$

At $t = T$, the maximum shortage (backorder) level J is

$$J = |I^2(T)| = \alpha(\Theta - t) + \beta \left(\frac{n}{m+n}\right) (\Theta^{\frac{m+n}{n}} - t^{\frac{m+n}{n}})$$

……(11)

6. COST ANALYSIS

The total inventory cost per cycle comprises five components: ordering, purchasing, holding, deterioration, and shortage costs. Average total inventory cost is obtained by dividing the cycle cost by the cycle length.

6.1 Ordering Cost (OC)

The ordering cost is a fixed charge incurred once per replenishment cycle, regardless of the quantity ordered,

$$OC = C_0$$

6.2 Purchasing Cost (PC)

The order quantity Q per cycle equals the sum of the initial stock E and the maximum backorder J as follows:

$$Q = E + J = I_1(0) + |I_2(T)|$$

$PC = C_p \cdot Q = C_p \cdot (E + J)$ (by putting values from equation (8) and (11) we get this value)

6.3 Holding Cost (HC)

The holding cost is incurred only during the positive inventory period ($0 \le t \le \Theta$):

$$HC = C_H \int_0^{\Theta} I_1(t) dt$$

Substituting the expression for $I_1(t)$ from equation (7) we get,

$$HC = C_H \int_0^{\Theta} [e^{-\gamma \frac{n}{m} t^{(\frac{m}{n})}} \int_t^{\Theta} \left(\alpha + \beta s^{\frac{m}{n}}\right) e^{\gamma \frac{n}{m} s^{\frac{m}{n}}} ds] dt$$

6.4 Deterioration Cost (DC)

The number of units that deteriorate during the positive inventory period equals the difference between the total demand and net change in inventory. The deterioration cost is

$$\boldsymbol{DC = C_D \int_0^{\theta} \lambda(t)\, I_1(t) \text{dt}}$$

Putting value from equation (7),

$$\boldsymbol{DC = C_D \int_0^{\theta} \lambda(t)\, [e^{-\gamma \frac{n}{m} t^{(\frac{m}{n})}} \int_t^{\Theta} \left(\alpha + \beta s^{\frac{m}{n}}\right) e^{\gamma \frac{n}{m} s^{\frac{m}{n}}} ds] \text{dt}}$$

6.5 Shortage Cost (SC)

The shortage cost is incurred during the backlogging period($\Theta \le t \le T$):

The shortage cost is only applicable during the shortage period.

$$\boldsymbol{SC = \int_{\theta}^{T} |I_2(t)|}\ \text{dt}$$

Where, $\boldsymbol{I_2(t) = \alpha(\theta - t) + \beta \frac{n}{m+n} (\theta^{\frac{m+n}{n}} - t^{\frac{m+n}{n}})}$

Since, $\boldsymbol{I_2(t) \le 0, |I_2(t)| = -|I_2(t)|}$

$$\boldsymbol{SC = C_s \int_{\theta}^{T} \left|\alpha(\theta - t) + \beta \frac{n}{m+n} (\theta^{\frac{m+n}{n}} - t^{\frac{m+n}{n}})\right|} \text{dt}$$

6.6 Total Inventory Cost per Cycle (TC)

$$TC = OC + PC + HC + DC + SC$$

$$TC = C_0 + C_p(E + J) + HC + DC + SC$$

6.7 Total Inventory Cost per Unit Time (TIC)

Total inventory cost per unit time is given by, $TIC = TC / T$

The objective is to minimize the TIC with respect to the two decision variables Θ and T.

6.8 Optimization Conditions

The optimal values of Θ and T are obtained by solving the system of equations:

$$\partial TIC/\partial\Theta = 0 \quad and \quad \partial TIC/\partial T = 0$$

The total minimum cost per unit time TIC (T, Θ) satisfy by sufficient condition,

$\frac{\partial^2 \mathrm{TIC}}{\partial^2 \Theta} > 0$

$\frac{\partial^2 \mathrm{TIC}}{\partial^2 \mathrm{T}} > 0$

$\frac{\partial^2 \mathrm{TIC}}{\partial^2 \Theta} * \frac{\partial^2 \mathrm{TIC}}{\partial^2 \mathrm{T}} - \frac{\partial^2 \mathrm{TIC}}{\partial \Theta\, \partial \mathrm{T}} > 0$

After solving this, we get optimal values of Θ and T ,which are Θ^* and T^*

7. NUMERICAL RESULTS AND DISCUSSION

The baseline numerical evaluation uses,

α=5, β=1, γ=0.2, m=2, n=3, C_0=100, C_p=10, C_H=2, $\boldsymbol{C_D}$=3, and $\boldsymbol{C_s}$=5.

Thus, $D(t) = 5 + t^{(2/3)}$ and $\lambda(t) = 0.2t^{(-1/3)}$.

The policy is optimized over $\Theta > 0$ and $T > \Theta$. The numerical integration is evaluated over the positive-inventory and shortage intervals, and the nonlinear cost function is minimized subject to the cycle-feasibility constraint.

With these parameters, the fractional demand rate becomes and the deterioration rate becomes Setting Θ = 3 and T = 6 as initial estimates and solving using Python (Appendix 1), the evaluation obtains the following cost breakdown:

Initial Inventory $(E) = 27.9964$, Maximum Backorder $(J) = 23.1428$, Order Quantity $(Q) = 51.1392$, Ordering Cost $(OC) = 100$, Purchasing Cost $(PC) = 511.3917$, Holding Cost $(HC) = 72.0953$, Deterioration Cost $(DC) = 27.7567$, Shortage Cost $(SC) = 169.0049$, Total Cost $(TC) = 880.2486$, Total Inventory Cost $(TIC) = 146.7081$.

7.1 Optimized Baseline Solution

The optimization gives Θ*=1.2114 and T*=2.8253. The corresponding minimum average total inventory cost is TIC*=123.1397 per unit time. At this solution, the initial stock E is 8.5178 units and the maximum backorder J is 10.6309 units, giving an order quantity Q=E+J=19.1487 units. The cost-per-cycle components are ordering 100.0000, purchasing 191.4871, holding 9.5497, deterioration 4.9040, and shortage 41.9606; the total cycle cost is 347.9014.

$\Theta^* = 1.211425, T^* = 2.825257,$

Minimum $TIC = 123.139732$

The optimized baseline policy provides a feasible balance between stock availability and the accumulation of backorders. The positive-inventory interval ends at Θ*, after which demand is accumulated as a fully backlogged shortage until the next replenishment at T*. The purchasing term is driven by the combined initial stock and maximum backorder, whereas holding and deterioration costs arise only during the positive-inventory interval. The shortage cost depends on both the length of the shortage interval and the accumulated demand during that interval. Consequently, changing Θ or T affects several cost components simultaneously, which explains why the decision variables must be optimized jointly rather than selected independently.

8. SENSITIVITY AND SCENARIO ANALYSIS

A one-factor-at-a-time sensitivity analysis was conducted by changing each selected parameter by −20% and +20%, while holding the remaining parameters at their baseline values. For each perturbed parameter set, the decision variables were re-optimized; therefore, the reported comparison concerns the optimized average cost rather than the cost at a fixed policy. This analysis identifies which assumptions most strongly influence the minimum cost. The values are summarized in Table 2 and visualized in Fig. 6. To make the evaluation more extensive, four additional scenarios were evaluated by increasing one parameter at a time by 20% and re-optimizing Θ and T. The scenario results are reported in Table 3. These are controlled numerical experiments rather than independent real-world datasets; they assess parameter robustness within the assumptions of the deterministic model.

9. SPECIAL CASES

The demand and deterioration functions reduce to simpler forms under selected parameter values. When β=0, demand is constant at α. When $m/n = 1$, demand becomes linear in time. When $\gamma = 0$, deterioration is absent. These are limiting or simplified cases of the proposed formulation; the exact relationship to a named published model depends on the remaining assumptions and cost structure. (Table 4)

Table 2. Sensitivity of the optimized average total inventory cost to ±20% parameter changes.

Parameter	Base Value	−20% TIC	Base TIC	+20% TIC	% Change (−20%)	% Change (+20%)	Sensitivity
Base demand α	123.14	108.86	123.14	137.21	-11.60%	+11.43%	High
Demand coefficient β	123.14	119.69	123.14	126.45	-2.80%	+2.69%	Low
Deterioration γ	123.14	121.20	123.14	124.77	-1.58%	+1.32%	Low
Holding cost C_H	123.14	122.43	123.14	123.78	-0.58%	+0.52%	Low

Ordering cost Co	123.14	115.68	123.14	129.90	-6.06%	+5.49%	Moderate
Deterioration cost C_D	123.14	122.79	123.14	123.48	-0.29%	+0.28%	Low
Shortage cost Cs	123.14	119.80	123.14	125.82	-2.71%	+2.17%	Low
Purchase cost C_p	123.14	109.46	123.14	136.59	-11.11%	+10.92%	High
Fractional exponent m/n	0.667	124.90	123.14	122.30	+1.43%	-0.69%	Low

Table 3. Re-optimized policy under selected parameter scenarios.

Scenario	Θ*	T*	Minimum TIC	Order quantity Q
Baseline	1.2114	2.8253	123.1397	19.1487
Higher base demand (+20%)	1.1471	2.6843	137.2140	20.9497
Higher deterioration (+20%)	1.0691	2.7570	124.7668	18.6336
Higher shortage cost (+20%)	1.2934	2.7196	125.8173	18.6224
Higher holding cost (+20%)	1.1487	2.7931	123.7815	18.7712

Table 4. Simplified cases of the proposed demand and deterioration formulation.

Special Case	Parameter Values	Reduces To
Linear time demand	$m = 1, n = 1$	$D(t) = \alpha + \beta t$ (Dave & Patel [5], Sachan [6])
Constant demand	$\beta = 0$	$D(t) = \alpha$ (classical EOQ with deterioration)
No deterioration	$\gamma = 0$	Standard EOQ with fractional demand and backlogging
No shortages	$\Theta = T$	Single-phase fractional demand deteriorating model
Integer power demand	$m/n = integer$	Polynomial time-dependent demand models

10. GRAPHICAL ANALYSIS

1. Inventory Level over One Replenishment Cycle: This figure 1 shows that inventory changes from available stock to shortage during one cycle (Appendix 3).

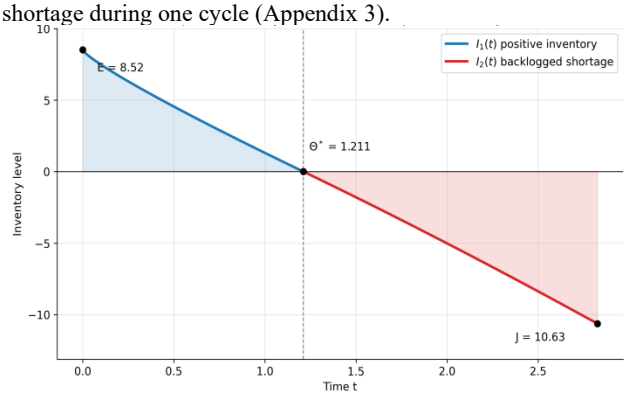


Figure 1: Inventory Level over One Replenishment cycle

2. Total Inventory Cost versus Cycle Time: This figure 2 shows that there is an optimum cycle time where the total cost is minimum (Appendix 3).

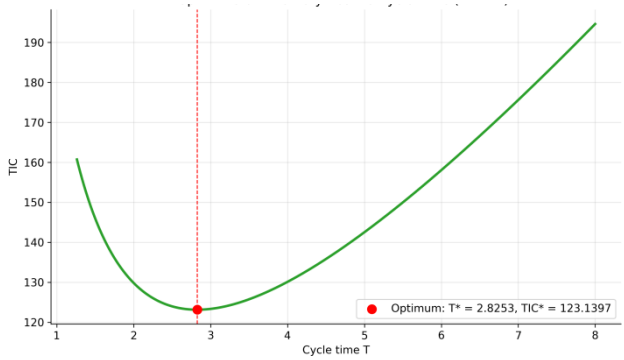


Figure 2: Total Inventory Cost versus Cycle Time

3. Total Inventory Cost versus Zero-Inventory Time: This figure 3 shows that there is an optimum time for allowing the inventory to reach zero (Appendix 3) .

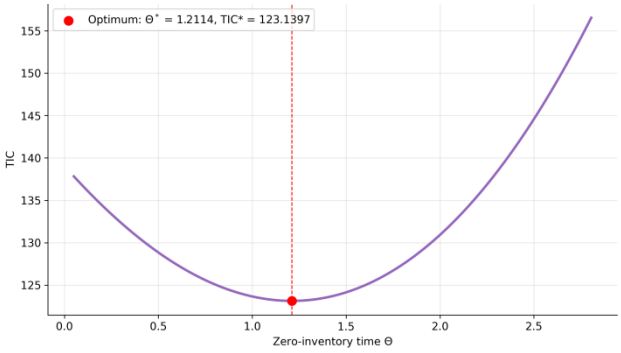


Figure 3: Total Inventory Cost versus Zero-Inventory Time

4. Cost Component Breakdown: This figure 4 compares the actual values of different inventory costs (Appendix 3)

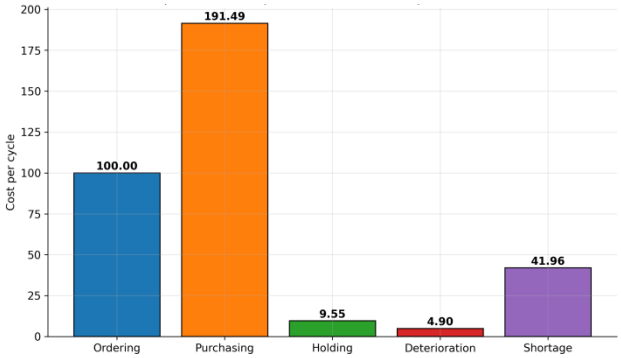


Figure 4: Cost Component Breakdown

5. Cost Component Proportion (figure 5) shows the percentage contribution of each cost to the total cost (Appendix 3).

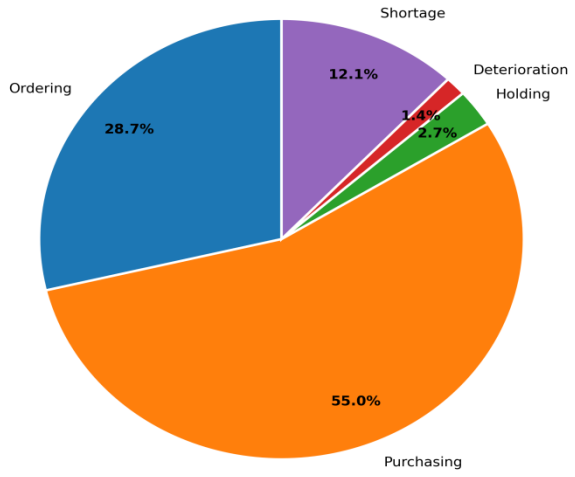


Figure 5: Cost Component Proportion

6. Sensitivity Analysis figure 6 shows which parameters have the greatest effect on the total inventory cost (Appendix 3.

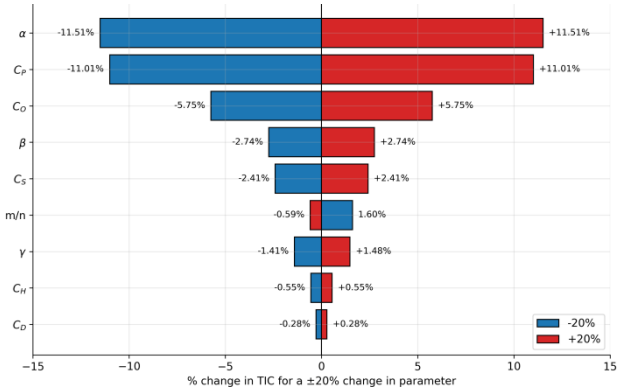


Figure 6: Sensitivity Analysis

7. Three-Dimensional Surface of Total Inventory Cost: This figure 7 shows the combined effect of cycle time and zero-inventory time and helps identify the optimal combination (Appendix 3).

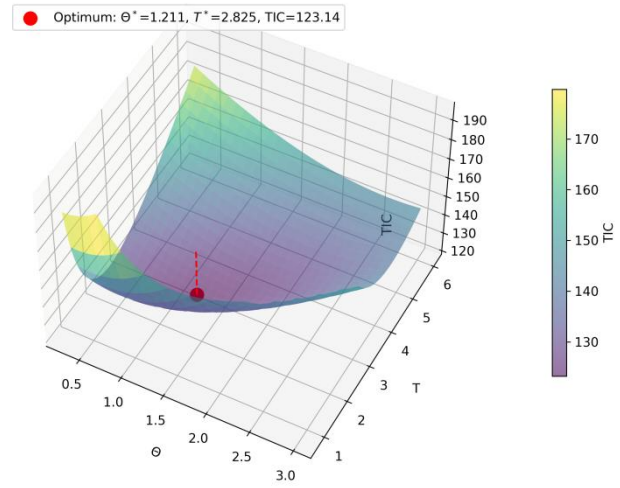


Figure 7: Three-Dimensional Surface of Total Inventory Cost

8. Demand and deterioration rates versus time: In this figure 8 the demand increases with time whereas the deterioration rate decreases with time for the selected parameters (Appendix 3).

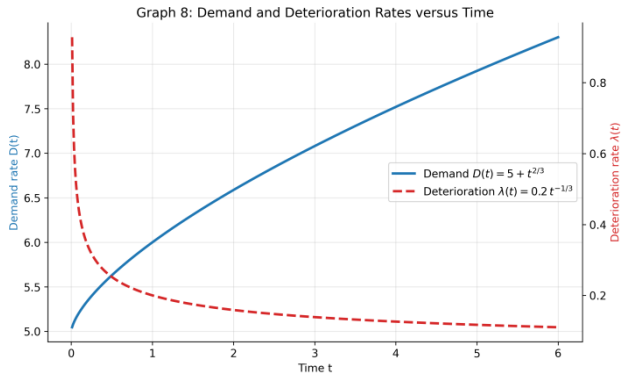


Figure 8. Demand and deterioration rates versus time

11. CONCLUSION

A deterministic single-item inventory model has been formulated for deteriorating items with fractional time-dependent demand, time-dependent deterioration, and fully backlogged shortages. The inventory dynamics are described separately for the positive-stock and shortage phases, and the

cost function includes ordering, purchasing, holding, deterioration, and shortage costs. For the baseline parameter set, numerical optimization gives $\Theta^* = 1.2114, T^* = 2.8253$, and a minimum average total inventory cost of 123.1397 per unit time. The sensitivity analysis shows how the optimized cost responds to changes in demand and cost parameters, providing a more informative assessment than a single numerical example. The results remain conditional on the deterministic demand specification, complete backlogging, zero lead time, and the selected parameter values. Future work may consider partial backlogging, stochastic demand, uncertain deterioration, multiple items, and empirical calibration using operational data.

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13. APPENDIX

Supplementary Material: The complete Python scripts used to generate Figures 1–8 , cost and optimal solution are available at: https://docs.google.com/document/d/11k7HA5NfdJjoSCUTS-4IEfGvm_BTiFew/edit?usp=drive_link&ouid=109611865044650721393&rtpof=true&sd=true