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Connectedness of Solution Sets and a Sharp Bound for the Number of its Components in Polynomial Vector Optimization

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ABSTRACT

This paper shows that, for vector optimization problems, the set of Karush-Kuhn-Tucker (KKT) points and strong KKT points can be characterized as sets of stationary points and proper stationary points, respectively, provided that the Mangasarian-Fromovitz constraint qualification (MFCQ) holds everywhere in the convex constraint set. As a consequence, by exploiting properties of semi-algebraic sets, it has been shown, via a direct approach, that the sets of stationary points, proper stationary points, and weak Pareto solutions all have finitely many connected components. This result holds for vector optimization problems in which both the constraint set and all objective function components are defined by polynomial functions, assuming MFCQ is satisfied everywhere in the convex constraint set. Moreover, if the set of proper stationary points is dense in the Pareto solution set, then the Pareto solution set itself also has finitely many connected components. Under convexity and the stated constraint qualification, an explicit upper bound on the number of connected components is derived. These bounds are significantly sharper than those previously reported in the literature. Finally, the connectedness structure of solution sets for special cases, including polynomial vector optimization and linear fractional vector optimization problems under polyhedral convex constraint sets, is explored.

General Terms

Convex optimization, Optimality

Keywords

polynomial vector optimization, semi-alebraic set, KKT points, constraint qualifications, connectedness

1. INTRODUCTION

The main motivation of this paper is to provide a different approach to studying the connectedness structure of solution sets of polynomial vector optimization problems, as established in [11] and [12]. Using a scalarization method and certain properties of semi-algebraic sets, Huong et al. [12] proved that both the proper Pareto solution set and the weak Pareto solution set of a vector variational inequality (VVI), where the convex constraint set is defined by polynomial functions and all components of the underlying operators are polynomial functions, have finitely many connected components, provided that the Mangasarian–Fromovitz constraint qualification holds at every point of the constraint set. Furthermore, if the proper Pareto solution set is dense in the Pareto solution set, then the latter also has finitely many connected components. As an application, similar results were established for vector optimization problems.

In [11], the authors made a similar attempt to study the connectedness structure of solution sets of a VVI where the constraint set is polyhedral convex and the underlying operators are polynomial functions. The results obtained in [11] provide affirmative answers to certain open problems in the general setting, without requiring monotonicity of the underlying operators.

For an introduction to the notion of vector variational inequalities, one refer to the pioneering work of Giannessi [8]. The relationship between VVI and vector optimization problems is well known (see [15]) and has attracted significant attention from researchers; see, for example, [11, 12, 9, 21, 20] and the references therein. The survey article [20] provides a comprehensive discussion of results on vector variational inequalities and vector optimization problems established in [11] and [12]. In [12], one of the key arguments in the proof of the main theorem is that the normal cone to the convex constraint set coincides with the Clarke normal cone [5, p. 51], which is the negative dual cone of the Clarke tangent cone. In this setting, the Clarke tangent cone admits an explicit representation (see [1, Remark, p. 151]). Indeed, the approach adopted in [12] can be viewed as a modification of the method developed earlier by Huong et al. [11]. The outline of the remainder of this paper is as follows. In Section 2, some basic notions and tools from semi-algebraic geometry are collected. In Section 3, the fundamental concepts of vector optimization problems, including KKT points and strong KKT points, are recalled. The relationship between KKT (resp., strong KKT) points and stationary (resp., proper stationary) points under the Mangasarian–Fromovitz constraint qualification (MFCQ) is investigated.

As a byproduct, by exploiting properties of semi-algebraic sets, it is shown that the sets of stationary points, proper stationary points, and weak Pareto solutions of a vector optimization problem where

the convex constraint set is defined by polynomial functions and all objective function components are polynomial have finitely many connected components, provided that MFCQ holds at every point of the constraint set. These results play a crucial role in deriving significantly sharper explicit upper bounds on the number of such components (cf. [10]).
In Section 4, a study of polynomial vector optimization and linear fractional vector optimization problems under polyhedral convex constraint sets as special cases is carried out, and the connectedness structure of their solution sets is considered. Finally, Section 5 concludes the paper and discusses directions for future research.

2. PRELIMINARIES

DEFINITION 1. (see [13]) A topological space X is called *connected* if it cannot be represented as the union of two disjoint, nonempty open subsets. A nonempty subset $A \subseteq X$ is said to be a *connected component* of X if A is connected in the subspace topology and is maximal with respect to this property, that is, A is not properly contained in any other connected subset of X.

The following result will be useful for analyzing the connectedness structure of the solution sets in subsequent sections.

LEMMA 1. (see [11, Lemma 2.3] and [13, Theorem 20, p. 54]) *Let K be a subset of a topological space X, and let $\overline{K}$ denote its closure. If K has k connected components, then any set M satisfying $K \subseteq M \subseteq \overline{K}$ has at most k connected components.*

2.1 Semi-algebraic geometry

In this section, a brief review of some fundamental concepts and tools from semi-algebraic geometry is presented, which play a central role in this paper.
Let $\mathbb{R}[x_1, \ldots, x_n]$ denote the ring of all real polynomials in variables $x_1, \ldots, x_n$.

DEFINITION 2. (see [3, Definition 2.1.4]) A subset of $\mathbb{R}^n$ is called *semi-algebraic* if it can be expressed as a finite union of finite intersections of sets of the form

$$\{x \in \mathbb{R}^n \mid f_{i,j}(x) \ast_{i,j} 0\},$$

where each $f_{i,j} \in \mathbb{R}[x_1, \ldots, x_n]$ and each relation symbol $\ast_{i,j}$ is either $<$ or $=$, for $i = 1, \ldots, s$ and $j = 1, \ldots, r_i$.

It is worth noting that the semi-algebraic subsets of $\mathbb{R}^n$ form the smallest collection of sets that contains all subsets of the type

$$\{x \in \mathbb{R}^n \mid f(x) < 0\},$$

where $f \in \mathbb{R}[x_1, \ldots, x_n]$, and is closed under finite unions, finite intersections, and complements.
Moreover, it follows (see [3, Proposition 2.1.8]) that any semi-algebraic subset of $\mathbb{R}^n$ can be represented as a finite union of sets of the form

$$\{x \in \mathbb{R}^n \mid f_1(x) = 0, \ldots, f_l(x) = 0,\ g_1(x) < 0, \ldots, g_m(x) < 0\},$$

where $f_1, \ldots, f_l, g_1, \ldots, g_m \in \mathbb{R}[x_1, \ldots, x_n]$.

Examples. Semi-algebraic subsets of $\mathbb{R}$ are precisely finite unions of points and (possibly unbounded) open intervals. In higher dimensions, standard examples include open balls, closed balls, and spheres in the Euclidean space $\mathbb{R}^n$.

Non-examples. The following sets are not semi-algebraic:

(i) $\{(x, y) \in \mathbb{R}^2 \mid y = e^x\}$,
(ii) $\{(x, y) \in \mathbb{R}^2 \mid \exists\, n \in \mathbb{N}$ such that $y = nx\}$.

2.2 Projection of Semi-algebraic Sets

By definition, the class of semi-algebraic sets is closed under finite unions, finite intersections, and complementation. In addition, an important structural property is that this class is also preserved under projections; see, for instance, [3, Theorem 2.2.1].

THEOREM 1 **TARSKI–SEIDENBERG**. *Let S be a semi-algebraic subset of $\mathbb{R}^{n+1}$, and let $\Pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ denote the canonical projection onto the first n coordinates. Then the image $\Pi(S)$ is a semi-algebraic subset of $\mathbb{R}^n$.*

By applying Theorem 1 iteratively with respect to the coordinates, and by suitably reordering them if necessary, one obtains the following general stability result for semi-algebraic sets.

THEOREM 2. *Let S be a semi-algebraic subset of $\mathbb{R}^{n+m}$. Let $\Phi : \mathbb{R}^{n+m} \to \mathbb{R}^n$ and $\Psi : \mathbb{R}^{n+m} \to \mathbb{R}^m$ denote the canonical projections onto the first n and the last m coordinates, respectively; that is,*

$$\Phi(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m}) = (x_1, \ldots, x_n),$$
$$\Psi(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m}) = (x_{n+1}, \ldots, x_{n+m}).$$

for every $x = (x_1, \ldots, x_{n+m}) \in \mathbb{R}^{n+m}$. Then $\Phi(S) \subseteq \mathbb{R}^n$ and $\Psi(S) \subseteq \mathbb{R}^m$ are semi-algebraic sets.

Recall the notion of semi-algebraic connectedness, which in fact agrees with the usual topological notion of connectedness in real algebraic geometry.

DEFINITION 3. (see [3, Definition 2.4.2]) A semi-algebraic set $A \subseteq \mathbb{R}^n$ is said to be *semi-algebraically connected* if for any pair of semi-algebraic subsets $F_1, F_2 \subseteq A$ that are closed in the relative topology, mutually disjoint, and satisfy $A = F_1 \cup F_2$, one necessarily has $F_1 = A$ or $F_2 = A$.

The following theorem establishes that semi-algebraic connectedness is equivalent to the usual topological connectedness for subsets of $\mathbb{R}^n$.

THEOREM 3. (see [3, Theorem 2.4.5]) *A semi-algebraic subset $A \subseteq \mathbb{R}^n$ is semi-algebraically connected if and only if it is connected in the usual topological sense. Moreover, every semi-algebraic set (and hence every algebraic subset of $\mathbb{R}^n$) admits only finitely many connected components, each of which is itself semi-algebraic.*

An explicit bound on the number of connected components of a semi-algebraic set can be obtained from the following result (see, e.g., [3] and [2, Exercise 4.4.6]).

THEOREM 4. *Let $f_1, \ldots, f_l, g_1, \ldots, g_m \in \mathbb{R}[x_1, \ldots, x_n]$ be polynomials of degree at most d. Consider the semi-algebraic set*

$$S = \{x \in \mathbb{R}^n \mid f_1(x) = 0, \ldots, f_l(x) = 0,\ g_1(x) \le 0, \ldots, g_m(x) \le 0\}.$$

Then the number of connected components of S does not exceed

$$((m + 1)d + 1)\,(2(m + 1)d + 1)^n.$$

3. VECTOR OPTIMIZATION PROBLEMS AND KKT POINTS

Denote by $\langle x, y\rangle$ the standard inner product of two vectors x, y in a Euclidean space. For a matrix M, the notation $M^\perp$ represents its transpose.

In this paper, the following vector optimization problem (VP) is discussed:

$$\begin{cases} \text{minimize} & f(x) = (f_1(x), \ldots, f_m(x)) \\ \text{subject to} & x \in \Omega, \end{cases} \tag{VP}$$

where $f := (f_1, \ldots, f_m) : \mathbb{R}^n \to \mathbb{R}^m$ is a vector-valued function, and Ω is a nonempty, closed, and convex subset of $\mathbb{R}^n$. In many applications, the feasible set Ω associated with problem (VP) admits the representation

$$\Omega = \{x \in \mathbb{R}^n \mid g_i(x) \le 0,\ h_j(x) = 0,\ i = 1, \ldots, k,\ j = 1, \ldots, p\}, \tag{1}$$

where $g_i : \mathbb{R}^n \to \mathbb{R}$ and $h_j : \mathbb{R}^n \to \mathbb{R}$ are real-valued functions.
When $m = 1$, problem (VP) reduces to a scalar optimization problem. In general, allow $k \ge 0$ and $p \ge 0$; if $k = 0$ (resp., $p = 0$), then no inequality (resp., equality) constraints are present.
Unless stated otherwise, all functions involved in (VP) are assumed to be continuously differentiable throughout the paper. For any feasible point $x \in \Omega$, define the active index set by

$$I(x) = \{i \in \{1, \ldots, k\} \mid g_i(x) = 0\},$$

which corresponds to the set of active (or binding) inequality constraints at x.
Let $\mathbb{R}^n_+$ and $\mathbb{R}^n_{++}$ denote the nonnegative orthant and its interior in $\mathbb{R}^n$, respectively.
Now recall two standard notions of minimality in vector optimization.

DEFINITION 4. *A feasible point $x \in \Omega$ is called an* efficient point *(or* Pareto optimal solution*) of* (VP) *if there does not exist any $y \in \Omega$ such that*

$$f(y) - f(x) \in -\mathbb{R}^m_+ \setminus \{0\}.$$

It is called a weakly efficient point *(or* weak Pareto optimal solution*) of* (VP) *if there is no $y \in \Omega$ such that*

$$f(y) - f(x) \in -\mathbb{R}^m_{++}.$$

It is well known that the existence of Lagrange multipliers at a feasible point $x \in \Omega$ does not, in general, guarantee that x is an efficient or weakly efficient solution of (VP). Consequently, there may exist feasible points that are not optimal, yet admit associated Lagrange multipliers.
Let W denote the collection of all tuples $(x, \tau, \lambda, \mu) = (x_1, .., x_n, \tau_1, .., \tau_m, \lambda_1, .., \lambda_k, \mu_1, .., \mu_p)$ such that $x \in \Omega$ and (τ, λ, μ) is a corresponding set of Lagrange multipliers at x. More precisely, define

$$\begin{aligned} W = \Big\{ &(x, \tau, \lambda, \mu) \in \mathbb{R}^{n+m+k+p} \ \Big| \\ &x \in \Omega, \\ &\sum_{r=1}^m \tau_r \nabla f_r(x) + \sum_{i=1}^k \lambda_i \nabla g_i(x) \\ &\quad + \sum_{j=1}^p \mu_j \nabla h_j(x) = 0, \\ &\lambda_i g_i(x) = 0 \text{ for } i = 1, \ldots, k, \\ &\tau_r \ge 0 \text{ for } r = 1, \ldots, m, \\ &\lambda_i \ge 0 \text{ for } i = 1, \ldots, k, \\ &\|(\tau, \lambda, \mu)\| > 0 \Big\}. \end{aligned}$$

If each g_i and h_j belongs to $\mathbb{R}[x_1, \ldots, x_n]$, then the feasible set Ω is described by k polynomial inequalities and p polynomial equalities in the variables $x_1, \ldots, x_n$, and is therefore a semi-algebraic subset of $\mathbb{R}^n$.
Furthermore, if all objective functions f_r, $r = 1, \ldots, m$, are also polynomials in $\mathbb{R}[x_1, \ldots, x_n]$, then the set W is defined by $2k+m$ polynomial inequalities, $p + n + k$ polynomial equations, together with one strict polynomial inequality. Consequently, W is a semi-algebraic subset of $\mathbb{R}^{n+m+k+p}$. Using Theorem 2, the set $\Phi(W)$ can be expressed as

$$\Phi(W) = \{x \in \mathbb{R}^n \mid (x, \tau, \lambda, \mu) \in W\}.$$

It follows that $\Phi(W)$ is also a semi-algebraic set, where $\Phi : \mathbb{R}^{n+m+k+p} \to \mathbb{R}^n$ denotes the projection onto the first n coordinates.
Hence, it is concluded that $\Phi(W)$, the collection of all feasible points, admitting a Lagrange multiplier is semi-algebraic.

Recall that for any feasible point x of (VP), a Karush–Kuhn–Tucker (KKT) multiplier associated with x is a triplet (τ, λ, μ) satisfying $(x, \tau, \lambda, \mu) \in W$ and $\|\tau\| > 0$. In this case, x is referred to as a KKT point.
Let W_{KKT} denote the subset of W consisting of all tuples (x, τ, λ, μ) for which x is a KKT point of (VP). It is straightforward to verify that W_{KKT} is itself semi-algebraic, and consequently, its projection $\Phi\big(W_{KKT}\big)$ is also semi-algebraic. Note that $\Phi\big(W_{KKT}\big)$ coincides with the set of all KKT points of (VP).
Formally, W_{KKT} can be written as

$$\begin{aligned} W_{KKT} = \Big\{ &(x, \tau, \lambda, \mu) \in \mathbb{R}^{n+m+k+p} \ \Big| \\ &x \in \Omega, \\ &\sum_{r=1}^m \tau_r \nabla f_r(x) + \sum_{i=1}^k \lambda_i \nabla g_i(x) \\ &\quad + \sum_{j=1}^p \mu_j \nabla h_j(x) = 0, \\ &\lambda_i g_i(x) = 0,\ \tau_r \ge 0\ (r = 1, \ldots, m), \\ &\lambda_i \ge 0\ (i = 1, \ldots, k),\ \|\tau\| > 0 \Big\} \\ = \Big\{ &(x, \tau, \lambda, \mu) \in W \ \Big| \ \|\tau\| > 0 \Big\}. \end{aligned} \tag{2}$$

For any feasible point x of (VP), a *strong* Karush–Kuhn–Tucker (KKT) multiplier associated with x is a triplet (τ, λ, μ) satisfying $(x, \tau, \lambda, \mu) \in W$ together with the strict positivity condition $\tau_r > 0$ for all $r = 1, \ldots, m$. In this case, x is referred to as a *strong* KKT point.
Let W_{SKKT} denote the collection of all tuples $(x, \tau, \lambda, \mu) \in W$ for which x is a strong KKT point of (VP). It follows that W_{SKKT} is a semi-algebraic set. Consequently, its projection $\Phi\big(W_{SKKT}\big)$ is also semi-algebraic, where $\Phi\big(W_{SKKT}\big)$ represents precisely the set of strong KKT points of (VP).
Conversely, by permuting the coordinate order and applying induction on the number of coordinates, together with Theorem 2, it follows that the set

$$\Psi(W) = \big\{(\tau, \lambda, \mu) \in \mathbb{R}^{m+k+p} \ \big| \ (x, \tau, \lambda, \mu) \in W\big\}$$

is also semi-algebraic. Here, $\Psi : \mathbb{R}^{n+m+k+p} \to \mathbb{R}^{m+k+p}$ denotes the natural projection onto the last $m + k + p$ coordinates. The set $\Psi(W)$ represents the collection of all Lagrange-Fritz John

multipliers associated with the vector optimization problem (VP) at feasible points where such multipliers exist. It is worth noting that $\Psi(W)$ need not be connected. This lack of connectedness may arise, for instance, if $\Phi(W)$, the set of feasible points, admitting Lagrange-Fritz John multipliers is itself disconnected.
However, when the constraints are polynomial, an important structural property emerges: since $\Psi(W)$ is semi-algebraic, it admits only finitely many connected components (see Theorem 3). Furthermore, an explicit upper bound on the number of these components is provided by Theorem 4.
For a fixed point $x \in \Omega$, we define the set

$$\begin{aligned} W(x) \; = \; \Bigg\{ (\tau, \lambda, \mu) \in \mathbb{R}^{m+k+p} \;\Big|\; & \sum_{r=1}^{m} \tau_r \nabla f_r(x) + \sum_{i=1}^{k} \lambda_i \nabla g_i(x) \\ & \sum_{j=1}^{p} \mu_j \nabla h_j(x) = 0, \\ & \text{and } \lambda_i g_i(x) = 0,\ \tau_r \geqslant 0,\ r = 1, .., m,\ \lambda_i \geqslant 0, \\ & i = 1, .., k,\ \|(\tau, \lambda, \mu)\| > 0 \Bigg\}. \end{aligned}$$

Thus, $W(x)$ is described by $(n+k)$ polynomial equations, $(m+k)$ polynomial inequalities, and one strict polynomial inequality in the variables $(\tau, \lambda, \mu) = (\tau_1, .., \tau_m, \lambda_1, .., \lambda_k, \mu_1, .., \mu_p) \in \mathbb{R}^{m+k+p}$. Consequently, $W(x)$ is a semi-algebraic set.

Although $W(x)$ may be empty in general, it is nonempty whenever x is a weak minimum, since in that case Lagrange–Fritz John multipliers are guaranteed to exist. Hence, $W(x) \neq \emptyset$ for every x belonging to the set of weakly efficient solutions.

Moreover, if $(\tau, \lambda, \mu) \in W(x)$, then for any scalar $\rho > 0$, the scaled tuple $(\rho\tau, \rho\lambda, \rho\mu)$ also belongs to $W(x)$. This shows that $W(x)$ is unbounded. Further discussion on this property can be found in [4], [6], [7], [18] and the references therein.

It is worth highlighting that $W(x)$ is always a semi-algebraic set. This property does not depend on whether the feasible set Ω itself is semi-algebraic, nor does it require that the objective functions f_r be polynomial. In particular, regardless of whether the functions f_r, g_i, and h_j are polynomial or belong to a broader class of functions, the semi-algebraic nature of $W(x)$ remains intact. This highlights the inherent presence of semi-algebraic structures in optimization theory.

Concerning the topological structure of $W(x)$, its semi-algebraicity ensures that it admits only finitely many connected components. An explicit upper bound on the number of such components can be obtained from Theorem 4. On the other hand, since $W(x)$ is convex, it is necessarily connected; in fact, it is path-connected.

Now focus on the vector optimization problem (VP) under the assumption that all functions f_r, g_i, and h_j are continuously differentiable, without requiring them to be polynomial. If, in addition, each f_r, $r = 1, \ldots, m$, is convex and the feasible set Ω is convex, then (VP) is referred to as a *convex* vector optimization problem.

The sets of Pareto optimal solutions and weak Pareto optimal solutions for (VP) will be denoted by $\mathrm{SOL}(VP)$ and $\mathrm{SOL}^w(VP)$, respectively. It follows directly from the definitions that

$$\mathrm{SOL}^w(\mathrm{VP}) \subseteq \{x \in \Omega \mid W(x) \neq \varnothing\} = \Phi(W).$$

The objective of this section is to identify conditions under which the above inclusion becomes an equality, or more generally, when $\mathrm{SOL}^w(\mathrm{VP})$ coincides with a suitable subset of $\Phi(W)$.

To derive the main results, the following theorem is needed. Note that here the feasible set Ω is assumed to be convex, but it does not need to be explicitly described by equality or inequality constraints.

THEOREM 5. ([15, Theorem 3.1, p. 749]) *Let Ω be a convex set and let $x \in \Omega$. Then:*

(*1*) *If x is a weak Pareto solution of (VP), then there exists a vector $\tau \in \mathbb{R}^m_+ \setminus \{\mathbf{0}\}$ such that*

$$\Big\langle \sum_{r=1}^{m} \tau_r \nabla f_r(x),\ y - x \Big\rangle \geq 0 \quad \textit{for all } y \in \Omega. \tag{3}$$

(*2*) *Conversely, if each function f_r, $r = 1, \ldots, m$, is convex and there exists $\tau \in \mathbb{R}^m_+ \setminus \{\mathbf{0}\}$ satisfying* (3)*, then x is a weak Pareto solution of (VP).*

(*3*) *Moreover, if all f_r, $r = 1, \ldots, m$, are convex and there exists $\tau \in \mathbb{R}^m_{++}$ such that* (3) *holds, then x is a Pareto solution of (VP).*

Now the notions of stationary and proper stationary points for the problem (VP), as proposed in [12], are defined. These concepts are inspired by analogous solution notions in vector variational inequality problems.

DEFINITION 5. Let Ω be a convex set and let $x \in \Omega$. Suppose that:

(1) there exists a vector $\tau \in \mathbb{R}^m_+ \setminus \{\mathbf{0}\}$ such that

$$\Big\langle \sum_{r=1}^{m} \tau_r \nabla f_r(x),\ y - x \Big\rangle \geq 0 \quad \text{for all } y \in \Omega, \tag{4}$$

then x is called a *stationary point* of (VP);

(2) there exists a vector $\tau \in \mathbb{R}^m_{++}$ satisfying (4), then x is called a *proper stationary point* of (VP).

REMARK 1. By comparing the above definitions with Theorem 5, it follows that the set of stationary points satisfies $\mathrm{SOL}^w(\mathrm{VP}) \subseteq \mathrm{Stat}(\mathrm{VP})$. Moreover, when all the objective functions f_r are convex, the set of proper stationary points is contained in the set of Pareto solutions, i.e., $\mathrm{Pr}(\mathrm{VP}) \subseteq \mathrm{SOL}(\mathrm{VP})$. Consequently, the following chain of inclusions holds:

$$\mathrm{Pr}(\mathrm{VP}) \subseteq \mathrm{SOL}(\mathrm{VP}) \subseteq \mathrm{SOL}^w(\mathrm{VP}) \subseteq \mathrm{Stat}(\mathrm{VP}),$$

where the first inclusion is valid under the convexity of all functions f_r.

DEFINITION 6. (see [17]) The Mangasarian–Fromovitz constraint qualification (MFCQ) is satisfied at a point $x \in \Omega$ if the following conditions hold:

(i) The set of gradient vectors $\{\nabla h_j(x) : j = 1, \ldots, p\}$ is linearly independent.

(ii) There exists a vector $v \in \mathbb{R}^n$ such that

$$\langle \nabla h_j(x), v \rangle = 0 \quad \text{for all } j = 1, \ldots, p,$$

and

$$\langle \nabla g_i(x), v \rangle < 0 \quad \text{for all } i \in I(x),$$

where $I(x) := \{i \mid g_i(x) = 0\}$ denotes the set of active constraints at x.

Impose the following assumption in order to analyze the relationship between the solution sets of (VP) and the sets of KKT and strong KKT points:

Assumption (A): The feasible set Ω is convex and the Mangasarian–Fromovitz constraint qualification (MFCQ) is satisfied at every point of Ω.

PROPOSITION 1. *Suppose that Assumption (A) holds for* (VP). *Then the following statements are valid:*

(i) $\mathrm{SOL}^w(\mathrm{VP}) \subseteq \Phi\big(W_{KKT}\big) = \mathrm{Stat}(\mathrm{VP})$. *Moreover, if each component function f_r is convex, then this inclusion becomes an equality.*

(ii) $\Phi\big(W_{SKKT}\big) = \Pr(\mathrm{VP})$.

PROOF. (i) Begin the proof by establishing the equality in part (i), namely

$$\Phi\big(W_{KKT}\big) = \mathrm{Stat}(\mathrm{VP}).$$

Recall that the feasible set is given by

$$\Omega = \{x \in \mathbb{R}^n \mid g_i(x) \le 0,\ h_j(x) = 0;\ i = 1, \dots, k,\ j = 1, \dots, p\}.$$

Since Ω is convex, its normal cone at a point $x \in \Omega$ is defined as

$$N_\Omega(x) = \{x^* \in \mathbb{R}^n \mid \langle x^*, y - x \rangle \leqslant 0 \ \ \forall y \in \Omega\}. \tag{5}$$

By Proposition 2.4.4 in [5], this normal cone coincides with the Clarke normal cone to Ω at x, which is the negative polar (dual) of the Clarke tangent cone $T_\Omega(x)$. Therefore,

$$N_\Omega(x) = \big(T_\Omega(x)\big)^* = \{x^* \in \mathbb{R}^n \mid \langle x^*, v \rangle \leqslant 0 \ \ \forall v \in T_\Omega(x)\}. \tag{6}$$

Since the Mangasarian–Fromovitz constraint qualification (MFCQ) holds at every point of Ω, the Clarke tangent cone admits the following explicit representation (see [1], Remark, p. 151): for each $x \in \Omega$,

$$T_\Omega(x) = \left\{ v \in \mathbb{R}^n \ \middle|\ \begin{array}{l} \langle \nabla g_i(x), v \rangle \leqslant 0 \ \ \forall i \in I(x), \\ \langle \nabla h_j(x), v \rangle = 0 \ \ \forall j = 1, \dots, p \end{array} \right\}.$$

It follows from (6) that a vector $x^* \in \mathbb{R}^n$ belongs to $N_\Omega(x)$ if and only if the inequality

$$\langle x^*, v \rangle \leqslant 0$$

holds for every v satisfying the above system of linear constraints.

$$\langle \nabla g_i(x), v \rangle \le 0 \ \forall i \in I(x), \langle \nabla h_j(x), v \rangle = 0 \quad \forall j = 1, \dots, p. \tag{7}$$

Applying Farkas' Lemma (by rewriting each equality constraint as a pair of inequalities; see Corollary 22.3.1 in [19]), it follows that $x^* \in N_\Omega(x)$ if and only if there exist multipliers $\lambda_i \geqslant 0$ for $i \in I(x)$ and $\mu_j \in \mathbb{R}$ for $j = 1, \dots, p$ such that

$$x^* = \sum_{i \in I(x)} \lambda_i \nabla g_i(x) + \sum_{j=1}^{p} \mu_j \nabla h_j(x).$$

Now, suppose that x is a KKT point. Then there exists $(x, \tau, \lambda, \mu) \in W$ with $\tau \in \mathbb{R}^m_+ \setminus \{\mathbf{0}\}$ satisfying

$$-\sum_{r=1}^{m} \tau_r \nabla f_r(x) = \sum_{i \in I(x)} \lambda_i \nabla g_i(x) + \sum_{j=1}^{p} \mu_j \nabla h_j(x).$$

From the preceding arguments, it follows that if x is a KKT point, then there exists $\tau \in \mathbb{R}^m_+ \setminus \{\mathbf{0}\}$ such that

$$-\sum_{r=1}^{m} \tau_r \nabla f_r(x) \in N_\Omega(x).$$

Equivalently, by the definition of the normal cone,

$$\left\langle \sum_{r=1}^{m} \tau_r \nabla f_r(x),\ y - x \right\rangle \geqslant 0 \quad \forall y \in \Omega.$$

Hence, by Definition 5, x is a stationary point of (VP), i.e., $x \in \mathrm{Stat}(\mathrm{VP})$. This shows that

$$\Phi\big(W_{KKT}\big) \subseteq \mathrm{Stat}(\mathrm{VP}).$$

Conversely, let $x \in \mathrm{Stat}(\mathrm{VP})$. Since Ω is convex, using the definition of stationary points and the arguments developed above, one can verify that x satisfies the KKT conditions. Therefore,

$$\mathrm{Stat}(\mathrm{VP}) \subseteq \Phi\big(W_{KKT}\big).$$

Combining both inclusions gives

$$\Phi\big(W_{KKT}\big) = \mathrm{Stat}(\mathrm{VP}).$$

The inclusion $\mathrm{SOL}^w(\mathrm{VP}) \subseteq \Phi\big(W_{KKT}\big)$ now follows directly from Remark 1 together with the equality established above. Finally, assume that each component function f_r is convex. Then, by part (ii) of Theorem 5, every KKT point is a weak Pareto solution of (VP). Hence,

$$\Phi\big(W_{KKT}\big) \subseteq \mathrm{SOL}^w(\mathrm{VP}).$$

Therefore,

$$\Phi\big(W_{KKT}\big) = \mathrm{SOL}^w(\mathrm{VP}).$$

This completes the proof of part (i). (ii) We now establish part (ii), namely,

$$\Phi\big(W_{SKKT}\big) = \Pr(\mathrm{VP}).$$

Assume first that x is a strong KKT point. Then there exists $\tau \in \mathbb{R}^m_{++}$ such that $(x, \tau, \lambda, \mu) \in W_{SKKT}$. Proceeding as in part (i), we obtain

$$\left\langle \sum_{r=1}^{m} \tau_r \nabla f_r(x),\ y - x \right\rangle \geqslant 0 \quad \forall y \in \Omega.$$

By Definition 5, this implies that x is a proper stationary point, i.e., $x \in \Pr(\mathrm{VP})$. Hence,

$$\Phi\big(W_{SKKT}\big) \subseteq \Pr(\mathrm{VP}).$$

Conversely, let $x \in \Pr(\mathrm{VP})$. Then there exists $\tau \in \mathbb{R}^m_{++}$ such that

$$\left\langle \sum_{r=1}^{m} \tau_r \nabla f_r(x),\ y - x \right\rangle \geqslant 0 \quad \forall y \in \Omega.$$

Since the Mangasarian-romovitz constraint qualification holds at every point of Ω, in particular at x, the argument used in part (i) shows that x satisfies the strong KKT conditions. Therefore,

$$\Pr(\mathrm{VP}) \subseteq \Phi\big(W_{SKKT}\big).$$

Combining the above inclusions gives

$$\Phi\big(W_{SKKT}\big) = \Pr(\mathrm{VP}).$$

This completes the proof of the proposition. □

3.1 Connectedness of Solution Sets in Polynomial Vector Optimization

Consider the problem (VP) in the case where all data are polynomial functions; that is, f_r, $g_i(x)$ and $h_j(x)$ belong to $\mathbb{R}[x_1,\ldots,x_n]$ for every $r=1,\ldots,m$, $i=1,\ldots,k$, and $j=1,\ldots,p$. The feasible set $\Omega\subseteq\mathbb{R}^n$ is described by k polynomial inequality constraints $g_i(x)\leq 0$ and p polynomial equality constraints $h_j(x)=0$, where $x=(x_1,\ldots,x_n)$.
Such a problem (VP) is referred to as a *polynomial vector optimization problem with polynomial constraints*, and we denote it by (pVP). The sets of Pareto optimal solutions, weak Pareto optimal solutions, stationary points, and properly stationary points of (pVP) are denoted by $\mathrm{SOL(pVP)}$, $\mathrm{SOL}^w\mathrm{(pVP)}$, $\mathrm{Stat(pVP)}$, and $\mathrm{Pr(pVP)}$, respectively.

A result concerning the connectedness of the solution sets of (pVP), originally established in [12], follows as an application of results derived for polynomial vector variational inequalities (see [12, Theorem 3.3]). In contrast to the approach used in [12], we present an alternative proof based on the Lagrange multiplier framework, making use of Proposition 1.

THEOREM 6. *Suppose that Assumption* (A) *holds for* (pVP). *Then the following assertions are valid:*

(i) *The sets* Stat(pVP) *and* Pr(pVP) *are semi-algebraic subsets of* $\mathbb{R}^n$. *Consequently, each of these sets admits only finitely many connected components, and every component is itself a semi-algebraic subset of* $\mathbb{R}^n$.

(ii) *If each objective function* f_r *is convex, then the set* SOL^w(pVP) *is semi-algebraic in* $\mathbb{R}^n$. *As a result, it can be decomposed into finitely many connected components, each of which is semi-algebraic.*

(iii) *Assume that all functions* f_r *are convex and that* Pr(pVP) *is dense in* SOL(pVP). *Then the set* SOL(pVP) *consists of a finite number of connected components.*

PROOF. In the case of (pVP), both the sets W_{KKT} and W_{SKKT} are semi-algebraic. Consequently, their images $\Phi\Big(W_{KKT}\Big)$ and $\Phi\Big(W_{SKKT}\Big)$ are also semi-algebraic sets. By Theorem 3, these image sets possess only finitely many connected components. Therefore, using Proposition 1, conclusions (i) and (ii) follow directly.

For part (iii), the argument follows similar reasoning as in [12]; however, we include it here for completeness. Since each f_r is convex, we have $\mathrm{Pr(pVP)}\subseteq\mathrm{SOL(pVP)}$. Moreover, the assumption that Pr(pVP) is dense in SOL(pVP) yields

$$\mathrm{Pr(pVP)}\subseteq\mathrm{SOL(pVP)}\subseteq\overline{\mathrm{Pr(pVP)}},\qquad (8)$$

where the closure is taken with respect to the standard Euclidean topology on $\mathbb{R}^n$.

From part (i), the set Pr(pVP) is semi-algebraic and thus has a finite number of connected components. Applying Lemma 1 to inclusion (8), we conclude that SOL(pVP) also has finitely many connected components. This completes the proof. □

REMARK 2. *The preceding theorem can be viewed as a consequence of Proposition 1, which plays a central role in describing the connection between the solution sets and the collections of KKT and strong KKT points. It is worth noting that the semi-algebraic nature of the solution sets considered here does not depend on convexity assumptions or any constraint qualification on* K*; see, for instance, [14, Remark 3.2]. However, the approach outlined in [14] does not yield any explicit upper estimate on the number of connected components of these sets. In contrast, incorporating convexity along with suitable constraint qualifications allows one to derive such bounds.*

Using Theorem 4, an explicit upper estimate for the number of connected components of the solution sets associated with polynomial vector optimization problems is derived. This bound significantly improves upon the one presented in [10, Theorem 2].

THEOREM 7. *Suppose that Assumption* (A) *holds for* (pVP). *Then the number of connected components* $\chi(S)$ *of a solution set* S *satisfies the following bounds:*

(i) *The number of connected components of the stationary set* $\chi(\mathrm{Stat(pVP)})$, *and similarly of the proper stationary set* $\chi(\mathrm{Pr(pVP)})$, *cannot exceed*

$$[(2k+m+2)d+1]\,[2(2k+m+2)d+1]^{n+m+k+p},\qquad (9)$$

where

$$d:=\max\Big\{\deg g_i,\ \deg h_j,\ \deg\frac{\partial f_r}{\partial x_l}\ \Big|\ {\scriptstyle r=1,\ldots,m;\ i=1,\ldots,k; \atop j=1,\ldots,p;\ l=1,\ldots,n}\Big\}+1.$$

(ii) *If, in addition, each function* f_r *is convex, then the number of connected components of the weak Pareto solution set* $\chi(\mathrm{SOL^w(pVP)})$ *also cannot exceed the number given in* (9).

PROOF. By Proposition 1, we have $\mathrm{Stat(VP)}=\Phi\big(W_{KKT}\big)$. From (2), it follows that a point $x\in W_{KKT}$ if and only if there exist multipliers $\tau\in\mathbb{R}^m$, $\lambda\in\mathbb{R}^k$, and $\mu\in\mathbb{R}^p$ such that

$$\begin{aligned}&\lambda_i\geqslant 0,\quad \lambda_i g_i(x)=0,\quad g_i(x)\leqslant 0,\quad i=1,\ldots,k,\\ &\tau_r\geqslant 0,\quad r=1,\ldots,m,\quad \|\tau\|>0,\\ &h_j(x)=0,\quad j=1,\ldots,p.\end{aligned}$$

and

$$\sum_{r=1}^m\tau_r\nabla f_r(x)+\sum_{i=1}^k\lambda_i\nabla g_i(x)+\sum_{j=1}^p\mu_j\nabla h_j(x)=0.\qquad (10)$$

Therefore, by (2), the set W_{KKT} consists of all points $(x,\tau,\lambda,\mu)\in\mathbb{R}^{n+m+k+p}$ satisfying the above conditions. The vector equation (10) yields n scalar equations given by

$$\sum_{r=1}^m\tau_r\frac{\partial f_r(x)}{\partial x_l}+\sum_{i=1}^k\lambda_i\frac{\partial g_i(x)}{\partial x_l}+\sum_{j=1}^p\mu_j\frac{\partial h_j(x)}{\partial x_l}=0,\quad l=1,\ldots,n.$$

Hence, W_{KKT} is a semialgebraic set described by $2k+m+1$ polynomial inequalities in $n+m+k+p$ variables, with degrees not exceeding d.

Applying Theorem 3, it follows that the projection $\Phi(W_{KKT})$ is also a semialgebraic set, where the mapping $\Phi:\mathbb{R}^{n+m+k+p}\to\mathbb{R}^n$ is defined by

$$\Phi(x,\tau,\lambda,\mu)=x.$$

Consequently, by Theorem 4, we obtain

$$\begin{aligned}\chi\big(\Phi(W_{KKT})\big)&\leqslant\chi(W_{KKT})\\ &\leqslant[(2k+m+2)d+1]\,[2(2k+m+2)d+1]^{n+m+k+p}.\end{aligned}$$

This completes the proof of part (i). If the functions f_r are convex, then the second part follows directly from Proposition 1 together with part (i). $\square$

4. SPECIAL CASES

Now, by employing arguments analogous to those developed in the previous section, we demonstrate that the results obtained in [11] for polynomial vector optimization and polynomial vector variational inequality problems under polyhedral convex constraint sets can be recovered in a similar framework. Once again, the role of KKT multipliers and strong KKT multipliers is fundamental. In particular, the relationship between stationary points and the sets of KKT points (namely, proper stationary points and strong KKT points) is effectively utilized.

4.1 Polynomial Vector Optimization under Polyhedral Convex Sets

Consider the problem (VP), where f_r are polynomials in $\mathbb{R}[x_1, \ldots, x_n]$ for all $r = 1, \ldots, m$, and the constraint set $\Omega \subseteq \mathbb{R}^n$ is a polyhedral convex set defined by k affine inequalities of the form ${a_i}^T x \leqslant b_i$ in the variables $x_1, \ldots, x_n$. That is,

$$\Omega = \{x \in \mathbb{R}^n \mid Ax \leqslant b\},$$

where $A \in \mathbb{R}^{k\times n}$ and $b \in \mathbb{R}^k$. The problem (VP) with such polynomial data is referred to as a *polynomial vector optimization problem under a polyhedral convex constraint set*, and is denoted by (VP_a). Clearly, (VP_a) is a special case of (pVP) with $g_i(x) = {a_i}^T x - b_i$ and without equality constraints h_j.
We denote the sets of Pareto minima, weak Pareto minima, stationary points, and proper stationary points of (VP_a) by $\mathrm{SOL}(\mathrm{VP}_a)$, $\mathrm{SOL}^w(\mathrm{VP}_a)$, $\mathrm{Stat}(\mathrm{VP}_a)$, and $\mathrm{Pr}(\mathrm{VP}_a)$, respectively.

As in the previous section, for a feasible point x of (VP_a), a Karush–Kuhn–Tucker (KKT) multiplier is a pair (τ, λ) such that $(x, \tau, \lambda) \in W^a$ with $\|\tau\| > 0$ (equivalently, $\tau_r > 0$ for all $r = 1, \ldots, m$), where

$$\begin{aligned} W^a = \Big\{ &(x,\tau,\lambda) \in \mathbb{R}^{n+m+k} \ \Big| \\ & x \in \Omega, \\ & \sum_{r=1}^{m} \tau_r \nabla f_r(x) + \sum_{i=1}^{k} \lambda_i a_i = 0, \\ & \lambda_i\, a_i^T x = 0, \\ & \tau_r \geqslant 0,\ r = 1,\ldots,m, \\ & \lambda_i \geqslant 0,\ i = 1,\ldots,k, \\ & \|(\tau,\lambda)\| > 0 \Big\}. \end{aligned}$$

A point x satisfying the above conditions is called a KKT point (respectively, a strong KKT point). Let W^a_{KKT} and W^a_{SKKT} denote the sets of all tuples $(x, \tau, \lambda) \in W^a$ corresponding to KKT points and strong KKT points of (VP_a), respectively. Then, these sets can be written as

$$\begin{aligned} W^a_{KKT} = \Big\{ &(x,\tau,\lambda) \in \mathbb{R}^{n+m+k} \ \Big| \\ & x \in \Omega, \\ & \sum_{r=1}^{m} \tau_r \nabla f_r(x) + \sum_{i=1}^{k} \lambda_i a_i = 0, \\ & \lambda_i\, a_i^T x = 0, \\ & \tau_r \geqslant 0,\ r = 1,\ldots,m, \\ & \lambda_i \geqslant 0,\ i = 1,\ldots,k, \\ & \|\tau\| > 0 \Big\}, \end{aligned}$$

and

$$\begin{aligned} W^a_{SKKT} = \Big\{ &(x,\tau,\lambda) \in \mathbb{R}^{n+m+k} \ \Big| \\ & x \in \Omega, \\ & \sum_{r=1}^{m} \tau_r \nabla f_r(x) + \sum_{i=1}^{k} \lambda_i a_i = 0, \\ & \lambda_i\, a_i^T x = 0, \\ & \tau_r > 0,\ r = 1,\ldots,m, \\ & \lambda_i \geqslant 0,\ i = 1,\ldots,k \Big\}. \end{aligned}$$

It is straightforward to verify that both W^a_{KKT} and W^a_{SKKT} are semialgebraic sets. Consequently, their projections $\Phi\big(W^a_{KKT}\big)$ and $\Phi\big(W^a_{SKKT}\big)$ are also semialgebraic. Here, $\Phi\big(W^a_{KKT}\big)$ and $\Phi\big(W^a_{SKKT}\big)$ represent the sets of KKT points and *strong* KKT points of (VP_a), respectively.
Next, a relationship between the solution sets of (VP_a) and the sets of KKT and strong KKT points is established.
It is evident that the polyhedral constraint set Ω is convex. However, due to the polyhedral nature of Ω, the Mangasarian–Fromovitz Constraint Qualification (MFCQ) may fail to hold at certain points of Ω, and therefore requires separate consideration. Nonetheless, arguments analogous to those used previously can be employed to derive the desired results.

PROPOSITION 2. *For* (VP_a)*, the following statements hold:*

(i)

$$\mathrm{SOL}^w(\mathrm{VP}_a) \subseteq \Phi\big(W^a_{KKT}\big) = \mathrm{Stat}(\mathrm{VP}_a).$$

If all the constituent functions f_r *of* f *are convex, then the above inclusion becomes an equality.*

(ii)

$$\Phi\big(W^a_{SKKT}\big) = \mathrm{Pr}(\mathrm{VP}_a).$$

Consequently, if all the constituent functions f_r *are convex, then* $\mathrm{SOL}^w(\mathrm{VP}_a)$ *is a semialgebraic subset of* $\mathbb{R}^n$*. In particular, it admits finitely many connected components, each of which is itself a semialgebraic subset of* $\mathbb{R}^n$*.*

PROOF. (i) First, establish the equality in part (i), namely,

$$\Phi\big(W^a_{KKT}\big) = \mathrm{Stat}(\mathrm{VP}_a).$$

Recall that

$$\Omega = \{x \in \mathbb{R}^n \mid Ax \leqslant b\},$$

where $A \in \mathbb{R}^{k\times n}$ and $b \in \mathbb{R}^k$. Equivalently,

$$\Omega = \{x \in \mathbb{R}^n \mid \langle a_i, x\rangle \leqslant b_i,\ i = 1, \dots, k\},$$

with $a_i \in \mathbb{R}^n$ and $b_i \in \mathbb{R}$. Clearly, Ω is a convex set. Therefore, the normal cone to Ω at a point $x \in \Omega$ is given by

$$N_\Omega(x) = \{x^* \in \mathbb{R}^n \mid \langle x^*, y - x\rangle \leqslant 0,\ \forall\, y \in \Omega\},$$

and $N_\Omega(x) = \emptyset$ if $x \notin \Omega$. Equivalently, $x^* \in N_\Omega(x)$ if and only if

$$\langle x^*, y\rangle \leqslant \langle x^*, x\rangle \quad \forall\, y \in \Omega.$$

By Theorem 22.3.1 of [19], $x^* \in N_\Omega(x)$ if and only if there exist multipliers $\lambda_i \geqslant 0$, $i = 1, \dots, k$, such that

$$\sum_{i=1}^{k} \lambda_i a_i = x^* \quad \text{and} \quad \sum_{i=1}^{k} \lambda_i b_i = \langle x^*, x\rangle. \tag{11}$$

Suppose that x is a KKT point. Then there exists $(x, \tau, \lambda) \in W^a_{KKT}$ such that

$$-\sum_{r=1}^{m} \tau_r \nabla f_r(x) = \sum_{i=1}^{k} \lambda_i a_i.$$

Comparing this with the first relation in (11), it follows that there exists $\tau \in \mathbb{R}^m_+ \setminus \{\mathbf{0}\}$ such that

$$-\sum_{r=1}^{m} \tau_r \nabla f_r(x) \in N_\Omega(x).$$

This implies that

$$\left\langle \sum_{r=1}^{m} \tau_r \nabla f_r(x),\ y - x \right\rangle \geqslant 0 \quad \forall\, y \in \Omega.$$

By Definition 5 of stationary points, conclude that $x \in \mathrm{Stat}(\mathrm{VP}_a)$. Hence,

$$\Phi\big(W^a_{KKT}\big) \subseteq \mathrm{Stat}(\mathrm{VP}_a).$$

Conversely, let $x \in \mathrm{Stat}(\mathrm{VP}_a)$. Since Ω is convex, using the definition of stationary points and the above arguments, it follows that x satisfies the KKT conditions. Therefore,

$$\mathrm{Stat}(\mathrm{VP}_a) \subseteq \Phi\big(W^a_{KKT}\big).$$

Thus,

$$\Phi\big(W^a_{KKT}\big) = \mathrm{Stat}(\mathrm{VP}_a).$$

The inclusion

$$\mathrm{SOL}^w(\mathrm{VP}_a) \subseteq \Phi\big(W^a_{KKT}\big)$$

follows from Remark 1 together with the above equality.
Now assume that all constituent functions f_r are convex. Then, by the second part of Theorem 5, every KKT point x is a weak Pareto minimum of (VP_a). Hence,

$$\Phi\big(W^a_{KKT}\big) \subseteq \mathrm{SOL}^w(\mathrm{VP}_a).$$

Consequently,

$$\Phi\big(W^a_{KKT}\big) = \mathrm{SOL}^w(\mathrm{VP}_a)$$

whenever all f_r are convex. This completes the proof of part (i).
(ii) Now to prove part (ii), namely,

$$\Phi\big(W^a_{SKKT}\big) = \mathrm{Pr}(\mathrm{VP}_a).$$

Suppose that x is a strong KKT point. Then there exists $\tau \in \mathbb{R}^m_{++}$ such that $(x, \tau, \lambda) \in W^a_{SKKT}$. Proceeding along the same lines as in the proof of part (i), results in

$$\left\langle \sum_{r=1}^{m} \tau_r \nabla f_r(x),\ y - x \right\rangle \geqslant 0 \quad \forall\, y \in \Omega.$$

By the definition of proper stationary points, it follows that $x \in \mathrm{Pr}(\mathrm{VP}_a)$. Hence,

$$\Phi\big(W^a_{SKKT}\big) \subseteq \mathrm{Pr}(\mathrm{VP}_a).$$

Conversely, assume that $x \in \mathrm{Pr}(\mathrm{VP}_a)$. Then there exists $\tau \in \mathbb{R}^m_{++}$ such that

$$\left\langle \sum_{r=1}^{m} \tau_r \nabla f_r(x),\ y - x \right\rangle \geqslant 0 \quad \forall\, y \in \Omega.$$

Since Ω is a polyhedral convex set, using arguments analogous to those in part (i), it follows that x satisfies the strong KKT conditions. Therefore, every point in $\mathrm{Pr}(\mathrm{VP}_a)$ is a strong KKT point, and consequently,

$$\mathrm{Pr}(\mathrm{VP}_a) \subseteq \Phi\big(W^a_{SKKT}\big).$$

Combining the above inclusions gives

$$\Phi\big(W^a_{SKKT}\big) = \mathrm{Pr}(\mathrm{VP}_a).$$

This completes the proof of the proposition.

□

4.2 Connectedness of solution Sets when the Constraint Set is Polyhedral Convex

The following results concerning the connectedness of solution sets of (VP_a) were established in [11] as an application of results on polynomial vector variational inequalities; see [11, Theorem 4.6]. For completeness, restate them below.

THEOREM 8. *For* (VP_a)*, the following statements hold:*

(i) *The sets* $\mathrm{Stat}(\mathrm{VP}_a)$ *and* $\mathrm{Pr}(\mathrm{VP}_a)$ *are semialgebraic subsets of* $\mathbb{R}^n$*. Consequently, each of these sets has finitely many connected components, and every connected component is itself a semialgebraic subset of* $\mathbb{R}^n$*.*

(ii) *If all the functions* f_r *are convex, then* $\mathrm{SOL}^w(\mathrm{VP}_a)$ *is a semialgebraic subset of* $\mathbb{R}^n$*. In particular, it has finitely many connected components, each of which is also semialgebraic.*

(iii) *If all the functions* f_r *are convex and the set* $\mathrm{Pr}(\mathrm{VP}_a)$ *is dense in* $\mathrm{SOL}(\mathrm{VP}_a)$*, then* $\mathrm{SOL}(\mathrm{VP}_a)$ *has finitely many connected components.*

PROOF. The assertions follow directly from Proposition 2 by employing arguments analogous to those used in the proof of Theorem 6 in Section 3. □

4.3 Linear Fractional Vector Optimization Problems under Polyhedral Convex Sets

Linear fractional vector optimization problems (LFVP) constitute an important class of vector optimization problems, with linear vector optimization problems forming a significant subclass. In this section, it is demonstrated that the results established in [11] for (LFVP) under polyhedral convex constraint sets originally derived as applications of polynomial vector variational inequalities can also be obtained via a direct approach. As before, the key tool is the interplay between stationary points and KKT-type points (namely, proper stationary points and strong KKT points).

Begin by recalling some basic properties of (LFVP). Consider the problem (VP), where $f_r : \mathbb{R}^n \to \mathbb{R}$, $r = 1, \dots, m$, are linear fractional functions of the form

$$f_r(x) = \frac{c_r^T x + \alpha_r}{d_r^T x + \beta_r},$$

where $c_r, d_r \in \mathbb{R}^n$ and $\alpha_r, \beta_r \in \mathbb{R}$. The feasible set $\Omega \subseteq \mathbb{R}^n$ is assumed to be a polyhedral convex set defined by k affine inequalities $a_i{}^T x \leqslant b_i$, $i = 1, \dots, k$, that is,

$$\Omega = \{x \in \mathbb{R}^n \mid Ax \leqslant b\},$$

with $A \in \mathbb{R}^{k \times n}$ and $b \in \mathbb{R}^k$. Assume that

$$d_r^T x + \beta_r > 0 \quad \forall\, r = 1, \dots, m,\ \forall\, x \in \Omega.$$

Let $f(x) = (f_1(x), \dots, f_m(x))$, and define

$$K = \{x \in \mathbb{R}^n \mid d_r^T x + \beta_r > 0,\ r = 1, \dots, m\}.$$

Then K is an open and convex set, $\Omega \subseteq K$, and f is continuously differentiable on K. The problem (VP) with the above structure is referred to as a *linear fractional vector optimization problem*, and is denoted by (VP$_f$), that is,

$$(\mathrm{VP}_f) \qquad \text{Minimize } f(x) \quad \text{subject to } x \in \Omega.$$

Denote the sets of Pareto minima, weak Pareto minima, stationary points, and proper stationary points of (VP$_f$) by $\mathrm{SOL}(\mathrm{VP}_f)$, $\mathrm{SOL}^w(\mathrm{VP}_f)$, $\mathrm{Stat}(\mathrm{VP}_f)$, and $\mathrm{Pr}(\mathrm{VP}_f)$, respectively.

According to [16], a point $x \in \mathrm{SOL}(\mathrm{VP}_f)$ if and only if there exists $\xi = (\xi_1, \dots, \xi_m) \in \mathbb{R}^m_{++}$ such that

$$\left\langle \sum_{r=1}^{m} \xi_r \left[(d_r^T x + \beta_r) c_r - (c_r^T x + \alpha_r) d_r \right],\ y - x \right\rangle \geqslant 0 \quad \forall\, y \in \Omega. \tag{12}$$

Similarly, $x \in \mathrm{SOL}^w(\mathrm{VP}_f)$ if and only if there exists $\xi = (\xi_1, \dots, \xi_m) \in \mathbb{R}^m_+ \setminus \{\mathbf{0}\}$ such that (12) holds.

In view of (12), it follows that

$$\mathrm{Pr}(\mathrm{VP}_f) = \mathrm{SOL}(\mathrm{VP}_f) \quad \text{and} \quad \mathrm{Stat}(\mathrm{VP}_f) = \mathrm{SOL}^w(\mathrm{VP}_f). \tag{13}$$

By arguments analogous to those used for (VP_a), one can define the sets of KKT points and strong KKT points for (VP_f) and establish a result similar to Proposition 2. Consequently, the sets $\mathrm{Pr}(\mathrm{VP}_f)$ and $\mathrm{Stat}(\mathrm{VP}_f)$ are semialgebraic subsets of $\mathbb{R}^n$.

Finally, invoking (13), we recover the following result, which was originally obtained in [12, Theorem 4.5] as an application of polynomial vector variational inequalities.

THEOREM 9. *For* (VP_f)*, the following statements hold:*

(i) *The Pareto solution set* $\mathrm{SOL}(\mathrm{VP}_f)$ *is a semialgebraic subset of* $\mathbb{R}^n$*. Consequently, it has finitely many connected components, and each connected component is itself a semialgebraic subset of* $\mathbb{R}^n$.

(ii) *The weak Pareto solution set* $\mathrm{SOL}^w(\mathrm{VP}_f)$ *is a semialgebraic subset of* $\mathbb{R}^n$*. In particular, it has finitely many connected components, each of which is also a semialgebraic subset of* $\mathbb{R}^n$.

REMARK 3. *Using arguments analogous to those in Theorem 7, one can derive explicit upper bounds on the number of connected components of the solution sets for polynomial vector optimization problems with polyhedral convex constraint sets, and similarly for linear fractional vector optimization problems.*

5. CONCLUSIONS

The primary objective of this paper has been to investigate the connectedness structure, an important topological property—of solution sets associated with polynomial vector optimization problems. To establish the main results previously obtained in [11] and [12], it is observed that the sets of KKT points and strong KKT points for such problems possess a semialgebraic structure. Furthermore, assuming that the Mangasarian-Fromovitz Constraint Qualification (MFCQ) holds at every point of the convex constraint set, we established that the set of KKT points (respectively, strong KKT points) coincides with the set of stationary points (respectively, proper stationary points) of the vector optimization problem. This observation plays a crucial role in deriving the connectedness properties of solution sets for polynomial vector optimization problems under MFCQ, as well as for polynomial and linear fractional vector optimization problems under polyhedral convex constraint sets.

In particular, it has been shown that the sets of stationary points, proper stationary points, and weak Pareto solutions of a polynomial vector optimization problem under the validity of MFCQ at each point of the convex constraint set have finitely many connected components. A sharper estimation of the number of connected components remains an important issue; such bounds can be derived using Theorem 6.Additionally, vector optimization problems and linear fractional vector optimization problems under polyhedral convex constraint sets have been examined as special cases within this framework.

Most importantly, the connectedness structure of (VP) has been analyzed through the use of KKT and strong KKT points. The approach adopted in this work is comparatively more direct and conceptually simpler than those employed in [11] and [12].

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