On Some Properties of Product of Fibonacci and Lucas Numbers

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ABSTRACT
This study presenting the properties of numbers that we get by multiplying Fibonacci and Lucas numbers. Namely we define recurrence relation $\mathcal{T}_{n+2} = 3\mathcal{T}_{n+1} - \mathcal{T}_n$, $n \geq 0$ $with$ $\mathcal{T}_0 = 0$, $\mathcal{T}_1 = 1$. We investigate some basic properties of product of Fibonacci and Lucas numbers such as the Binet formula, generating function, generalized identity. We shall use the Binet's formula and generating function for derivation. Also, we present its two cross two matrix representation.

Keywords
Fibonacci number, Lucas Number, Binet's formula, Generating function and Matrices.

1. INTRODUCTION
The well-known Fibonacci sequence [11], with $F_0 = 0, F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$, have many interesting interpretations, applications and generalizations. Lucas sequence [11], with $L_0 = 2, L_1 = 1$ and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$. Most of the authors introduced Fibonacci pattern-based sequences in many ways which are known as generalized Fibonacci sequences ([9]; [8]; [1]), Fibonacci-Like sequences ([15]; [7]; [19]), k-Fibonacci sequences [3] and k-Fibonacci-Like sequences ([14]; [18]).

([16]; [17]) presents a family of tridiagonal matrices given by:

$$M(n) = \begin{pmatrix} 3 & 1 & & & \\ 1 & 3 & 1 & & \\ & 1 & 3 & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & 3 \end{pmatrix}$$

Where $M(n)$ is $n \times n$. It is easy to show by induction that the determinants $|M(k)|$ are the Fibonacci numbers F_{2k+2}. [2], extend these results to construct families of tridiagonal matrices whose determinants generate any arbitrary linear subsequence $F_{\alpha k+\beta}$ or $L_{\alpha k+\beta}, k = 1, 2, \ldots$ of the Fibonacci or Lucas numbers.

Lu and Jiang [12], present product of Fibonacci and Lucas numbers $f_n = (F_n \times L_n)$. Also defined and present some determinant and permanent representations of introduce product of Fibonacci and Lucas numbers and complex factorization formulas for $f_n = (F_n \times L_n)$.

The recurrence relation for product of Fibonacci and Lucas numbers $\mathcal{T}_{n+2} = 3\mathcal{T}_{n+1} - \mathcal{T}_n$, $n \geq 1$ $with$ $\mathcal{T}_1 = 1, \mathcal{T}_2 = 3$. Wei, Zheng, Jiang and Shon [20] discuss the invertibility of the skew circulant and skew left circulant matrices involving the product of Fibonacci and Lucas numbers and present the determinant and inverse matrices by constructing the transformation matrices. The four kinds of norms and bounds for the spread of these matrices are given, respectively. More specifically, they study the invertibility, determinant, multiple norms, lower and upper bounds for the spread of these matrices, which are going to have potential to be useful for realistic application. This study introduce some properties of product of Fibonacci and Lucas numbers.

2. SEQUENCE OF PRODUCT OF FIBONACCI AND LUCAS NUMBERS
Definition: The product of Fibonacci and Lucas numbers $\{\mathcal{T}_n\}$, see [12, 20] are defined by the recurrence relation: For $n \geq 0$,

$$\boldsymbol{\mathcal{T}_{n+2} = 3\mathcal{T}_{n+1} - \mathcal{T}_n} \quad (2.1)$$

with the initial conditions $\boldsymbol{\mathcal{T}_0 = 0}$ and $\boldsymbol{\mathcal{T}_1 = 1}$.

Let τ_1 and τ_2 be the roots of the following characteristic equation,

$$\boldsymbol{t^2 - 3t + 1 = 0} \quad (2.2)$$

Associated to the recurrence relation (2.1).

Where $\boldsymbol{\tau_1 = \frac{3+\sqrt{5}}{2}, \tau_2 = \frac{3-\sqrt{5}}{2}}$ and $\boldsymbol{\tau_1\tau_2 = 1}$.

The first few numbers are given in the following table:

Table 1. Product of Fibonacci and Lucas numbers

0	0	1	2	3	4	5	6	7 …
F_n	0	1	1	2	3	5	8	13 …
L_n	2	1	3	4	7	11	18	29 …
$\boldsymbol{\mathcal{T}_n = (F_n \times L_n)}$	**0**	**1**	**3**	**8**	**21**	**55**	**144**	**377…**

3. PROPERTIES OF PRODUCT OF FIBONACCI AND LUCAS NUMBERS
3.1 Binet Formula
The Binet formula is also very important in Fibonacci and Lucas numbers theory. Now we can give the Binet formula for the product of Fibonacci and Lucas numbers.

Theorem 3.1: For $\boldsymbol{n \geq 0}$, $\boldsymbol{\mathcal{T}_n = \frac{\tau_1^n - \tau_2^n}{\tau_1 - \tau_2}}$ (3.1)

Proof: The theorem can be proved by mathematical induction on n.

Lemma 3.2: For any integer $\boldsymbol{n \geq 0}$,

$$\tau_1^{n+2} + \tau_1^n = 3\tau_1^{n+1} \text{ and } \tau_2^{n+2} + \tau_2^n = 3\tau_2^{n+1} \quad (3.2)$$

Proof: Since $\tau_1 and$ τ_2 are the roots of the characteristic equation (2.2), then

$$\tau_1^2 + 1 = 3\tau_1 \text{ and } \tau_2^2 + 1 = 3\tau_2 \quad (3.3)$$

now, multiplying both sides of these equations by τ_1^n and τ_2^n respectively, we obtain the desired result.

Theorem 3.2: For any integer $\boldsymbol{n \geq 1}$,

$$\mathcal{T}_{(n+1)} + \mathcal{T}_{(n-1)} = 3\mathcal{T}_n \qquad (3.4)$$

Proof: By using Eq. (3.1) in the left-hand side (LHS) of Eq. (3.4), and considering that $\tau_1^2 + 1 = 3\tau_1$ and $\tau_2^2 + 1 = 3\tau_2$, it is obtained

$$(LHS) = \left(\frac{\tau_1^{n+1} - \tau_2^{n+1}}{\tau_1 - \tau_2}\right) + \left(\frac{\tau_1^{n-1} - \tau_2^{n-1}}{\tau_1 - \tau_2}\right)$$

$$= \frac{\tau_1^n\left(\tau_1 + \frac{1}{\tau_1}\right) - \tau_2^n\left(\tau_2 + \frac{1}{\tau_2}\right)}{\tau_1 - \tau_2}$$

$$= 3\left(\frac{\tau_1^n - \tau_2^n}{\tau_1 - \tau_2}\right)$$

and, again by Eq. (3.1), this completes the proof.

Theorem 3.4: For any integer $\boldsymbol{n \geq 1}$,

$$\mathcal{T}_n^2 + \mathcal{T}_{n+1}^2 = \frac{3(\tau_1^{2n+2} + \tau_2^{2n+2}) - 4}{5} \qquad (3.5)$$

Proof: By using Binet's formula (3.1),

$$\mathcal{T}_n^2 + \mathcal{T}_{n+1}^2 = \left(\frac{\tau_1^n - \tau_2^n}{\tau_1 - \tau_2}\right)^2 + \left(\frac{\tau_1^{n+1} - \tau_2^{n+1}}{\tau_1 - \tau_2}\right)^2$$

$$= \frac{(\tau_1^{2n} + \tau_1^{2n+2} + \tau_2^{2n} + \tau_2^{2n+2} - 4)}{(\tau_1 - \tau_2)^2}$$

$$= \frac{\{\tau_1^{2n}(\tau_1^2 + 1) + \tau_2^{2n}(\tau_2^2 + 1) - 4\}}{(\tau_1 - \tau_2)^2}$$

$$= \frac{3(\tau_1^{2n+2} + \tau_2^{2n+2}) - 4}{5}$$

This completes the proof.

Theorem 3.5: ***(Catalan's identity)***

$$\mathcal{T}_n^2 - \mathcal{T}_{n+r}\mathcal{T}_{n-r} = \mathcal{T}_r^2 \qquad (3.6)$$

Proof: By using Binet's formula (3.1),

$$\mathcal{T}_n^2 - \mathcal{T}_{n+r}\mathcal{T}_{n-r}$$

$$= \frac{1}{(\tau_1 - \tau_2)^2}\left\{(\tau_1\tau_2)^n\left(\frac{\tau_1}{\tau_2}\right)^r + (\tau_1\tau_2)^n\left(\frac{\tau_2}{\tau_1}\right)^r - 2(\tau_1\tau_2)^n\right\}$$

$$= \frac{(\tau_1\tau_2)^n}{(\tau_1 - \tau_2)^2}\left\{\left(\frac{\tau_1}{\tau_2}\right)^r + \left(\frac{\tau_2}{\tau_1}\right)^r - 2\right\}$$

$$= \frac{(\tau_1\tau_2)^n}{(\tau_1 - \tau_2)^2}\left\{\frac{\tau_1^{2r} + \tau_2^{2r} - 2(\tau_1\tau_2)^r}{(\tau_1\tau_2)^r}\right\}$$

$$= (\tau_1\tau_2)^{n-r}\left(\frac{\tau_1^r - \tau_2^r}{\tau_1 - \tau_2}\right)^2$$

$$= \mathcal{T}_r^2$$

This completes the proof.

Theorem 3.6: ***(Cassini's identity or Simpson's identity)***

$$\mathcal{T}_n^2 - \mathcal{T}_{n+1}\mathcal{T}_{n-1} = 1 \qquad (3.7)$$

Proof: Taking *r*=1 in identity (3.6) and (3.7) the proof is completed.

Theorem 3.7: ***(d'ocagnes's Identity)*** For $\boldsymbol{m > n}$,

$$\mathcal{T}_m\mathcal{T}_{n+1} - \mathcal{T}_{m+1}\mathcal{T}_n = \mathcal{T}_{m-n} \qquad (3.8)$$

Proof: By using Eq. (3.1) in left hand side (LHS) of Eq. (3.8), and considering that $\tau_1\tau_2 = 1$, it is obtained

$$(LHS) = \left(\frac{\tau_1^m - \tau_2^m}{\tau_1 - \tau_2}\right)\left(\frac{\tau_1^{n+1} - \tau_2^{n+1}}{\tau_1 - \tau_2}\right) - \left(\frac{\tau_1^{m+1} - \tau_2^{m+1}}{\tau_1 - \tau_2}\right)\left(\frac{\tau_1^n - \tau_2^n}{\tau_1 - \tau_2}\right)$$

$$= \frac{\tau_1^m\tau_2^n(\tau_1 - \tau_2) + \tau_2^m\tau_1^n(\tau_1 - \tau_2)}{(\tau_1 - \tau_2)^2}$$

$$= \frac{\tau_1^m\tau_2^n - \tau_2^m\tau_1^n}{(\tau_1 - \tau_2)}$$

$$= \frac{\tau_1^{m-n} - \tau_2^{m-n}}{(\tau_1 - \tau_2)}$$

and, again by Eq. (3.1), the result is obtained.

Theorem 3.8: ***(Limit of the quotient of two consecutive terms)***
For $\boldsymbol{n \geq 2}$, $\lim_{n\to\infty}\left(\frac{\mathcal{T}_n}{\mathcal{T}_{n-1}}\right) = \tau_1 \qquad (3.9)$

Proof: By Binet's formula (3.1), we have

$$\lim_{n\to\infty}\left(\frac{\mathcal{T}_n}{\mathcal{T}_{n-1}}\right) = \lim_{n\to\infty}\left(\frac{\tau_1^n - \tau_2^n}{\tau_1^{n-1} - \tau_2^{n-1}}\right)$$

$$= \lim_{n\to\infty}\frac{1 - \left(\frac{\tau_2}{\tau_1}\right)^n}{\frac{1}{\tau_1} - \left(\frac{\tau_2}{\tau_1}\right)^n\frac{1}{\tau_2}}$$

and considering that $\lim_{n\to\infty}\left(\frac{\tau_2}{\tau_1}\right)^n = 0,\ since\ |\tau_2| < \tau_1$, Eq. (3.9) is obtained.

Theorem 3.9: For every integer *n*, $\mathcal{T}_{-n} = -\mathcal{T}_n$ (3.10)

Proof: By Binet's formula (3.1), we have

$$\mathcal{T}_{-n} = \frac{\tau_1^{-n} - \tau_2^{-n}}{\tau_1 - \tau_2}$$

$$= \frac{\left(\frac{1}{\tau_1^n} - \frac{1}{\tau_2^n}\right)}{\tau_1 - \tau_2}$$

$$= \frac{\tau_2^n - \tau_1^n}{\tau_1 - \tau_2}$$

$$= -\frac{\tau_1^n - \tau_2^n}{\tau_1 - \tau_2}$$

and, again by Eq. (3.1), the result is obtained.

Theorem 3.10: $\sum_{i=0}^{n-1} \mathcal{T}_i = \mathcal{T}_n - \mathcal{T}_{n-1} - 1$ (3.11)

Proof: The proof is clear by Binet's formula.

Theorem 3.11: For $n \geq 1$, $t^n = t\mathcal{T}_n - \mathcal{T}_{n-1}$ (3.12)

Proof: From the characteristic equation (2.2), we have

$$t^2 = 3t - 1 = t\,\mathcal{T}_2 - \mathcal{T}_1 \tag{3.13}$$

By induction on *n*, we get

$$t^{n+1} = t^n t = (t\mathcal{T}_n - \mathcal{T}_{n-1})t$$

$$= t^2\mathcal{T}_n - t\,\mathcal{T}_{n-1}$$

$$= (3t-1)\,\mathcal{T}_n - t\mathcal{T}_{n-1}$$

$$= (3\mathcal{T}_n - \mathcal{T}_{n-1})t - \mathcal{T}_n$$

$$= \mathcal{T}_{n+1}t - \mathcal{T}_n$$

Therefore, we have, $t^n = t\mathcal{T}_n - \mathcal{T}_{n-1}$

Theorem 3.12: ***(Generalized identity)*** For $n > m \geq k \geq 1$,

$$\mathcal{T}_m\mathcal{T}_n - \mathcal{T}_{m-k}\mathcal{T}_{n-k} = \mathcal{T}_k\mathcal{T}_{n-m+k} \tag{3.14}$$

Proof: By Binet's formula (3.1), we have

$$\mathcal{T}_m\mathcal{T}_n - \mathcal{T}_{m-k}\mathcal{T}_{n-k} = \left(\frac{\tau_1^m - \tau_2^m}{\tau_1 - \tau_2}\right)\left(\frac{\tau_1^n - \tau_2^n}{\tau_1 - \tau_2}\right) - \left(\frac{\tau_1^{m-k} - \tau_2^{m-k}}{\tau_1 - \tau_2}\right)\left(\frac{\tau_1^{n-k} - \tau_2^{n-k}}{\tau_1 - \tau_2}\right)$$

$$= \frac{\tau_1^m\tau_2^n\left(\tau_1^{-k}\tau_2^k - 1\right) + \tau_1^n\tau_2^m(\tau_1^k\tau_2^{-k} - 1)}{(\tau_1 - \tau_2)^2}$$

$$= \frac{\left(\tau_1^k - \tau_2^k\right)}{(\tau_1 - \tau_2)^2}\left(\frac{\tau_1^{n-m}}{\tau_1{}^k} - \frac{\tau_2^{n-m}}{\tau_2{}^k}\right)$$

$$= \frac{\left(\tau_1^k - \tau_2^k\right)}{(\tau_1 - \tau_2)}\left(\frac{\tau_1^{n-m+k} - \tau_2^{n-m+k}}{\tau_1 - \tau_2}\right)$$

$$= \mathcal{T}_k\mathcal{T}_{n-m+k}$$

This completes the proof.

Corollary 3.12.1: *(Catlan's identity).* If ***m=n*** in the generalized identity (3.14), we obtain,

$$\mathcal{T}_n^2 - \mathcal{T}_{n-k}\mathcal{T}_{n+k} = \mathcal{T}_k^2 \tag{3.15}$$

Corollary 3.12.2: *(Cassini's identity).* If *m*=*n and k*=1in the generalized identity (3.14), we obtain,

$$\mathcal{T}_n^2 - \mathcal{T}_{n-1}\mathcal{T}_{n+1} = 1 \tag{3.16}$$

Corollary 3.12.3: *(d'Ocagne's identity)*. If $n = m$, $m = n + 1$ and $k = 1$ in the generalized identity (3.14), we obtain, $\mathcal{T}_m\mathcal{T}_{n+1} - \mathcal{T}_n\mathcal{T}_{m+1} = \mathcal{T}_{m-n}$ (3.17)

Theorem 3.13: $\mathcal{T}_{m+n}\mathcal{T}_{m+t} - \mathcal{T}_m\mathcal{T}_{m+n+t} = \mathcal{T}_n\mathcal{T}_t$ (3.18)

Proof: By Binet's formula (3.1), the proof is clear.

3.2 Generating Function

In this section, we present generating function for product of Fibonacci and Lucas numbers.

Theorem 3.14: Let $G_{\mathcal{T}=F\times L}$ be the generating functions of the product of Fibonacci and Lucas numbers, then

$$G_{\mathcal{T}=F\times L} = \frac{t}{1-3t+t^2} \tag{3.19}$$

Proof: $G_{\mathcal{T}=F\times L} = \sum_{n=0}^{\infty}\mathcal{T}_n t^n$
$= \mathcal{T}_0 + \mathcal{T}_1 t + \mathcal{T}_2 t^2 + \sum_{n=3}^{\infty}\mathcal{T}_n t^n$
$= t + 3t^2 + \sum_{n=3}^{\infty}(3\mathcal{T}_{n-1} - \mathcal{T}_{n-2})t^n$
$= t + 3t^2 + \sum_{n=3}^{\infty}3\mathcal{T}_{n-1}t^n - \sum_{n=3}^{\infty}3\mathcal{T}_{n-2}t^n$
$= t + 3t^2 + t\sum_{n=3}^{\infty}3\mathcal{T}_{n-1}t^{n-1} - t^2\sum_{n=3}^{\infty}3\mathcal{T}_{n-2}t^{n-2}$
$= t + 3t^2 + 3t\sum_{n=2}^{\infty}\mathcal{T}_n t^n - t^2\sum_{n=1}^{\infty}\mathcal{T}_n t^n$
$= t + 3t^2 + 3t[\sum_{n=1}^{\infty}\mathcal{T}_n t^n - t] - t^2\sum_{n=1}^{\infty}\mathcal{T}_n t^n$
$= t + 3t^2 + 3t[G_{\mathcal{T}=F\times L} - t] - t^2 G_{\mathcal{T}=F\times L}$
Thus, $G_{\mathcal{T}=F\times L} = \frac{t}{1-3t+t^2}$
This completes the proof.

Theorem 3.15: For $\mathcal{p}, \mathcal{q} \in \mathbf{Z}$, we get
$$\sum_{n=0}^{\mathcal{p}+\mathcal{q}}\mathcal{T}_n t^{-n} = \frac{t}{t^{\mathcal{p}+\mathcal{q}}(t^2-3t+1)}\left(t^{\mathcal{p}+\mathcal{q}+1} - t\mathcal{T}_{\mathcal{p}+\mathcal{q}+1} + \mathcal{T}_{\mathcal{p}+\mathcal{q}}\right) \tag{3.20}$$
Proof: By the Binet's formula,
$\sum_{n=0}^{\mathcal{p}+\mathcal{q}}\mathcal{T}_n t^{-n} = \sum_{n=0}^{\mathcal{p}+\mathcal{q}}\left(\frac{\tau_1^n - \tau_2^n}{\tau_1-\tau_2}\right)t^{-n}$
$= \frac{1}{\tau_1-\tau_2}\sum_{n=0}^{\mathcal{p}+\mathcal{q}}\left\{\left(\frac{\tau_1}{t}\right)^n - \left(\frac{\tau_2}{t}\right)^n\right\}$
$= \frac{1}{\tau_1-\tau_2}\left\{\frac{1-\left(\frac{\tau_1}{t}\right)^{\mathcal{p}+\mathcal{q}+1}}{1-\frac{\tau_1}{t}} - \frac{1-\left(\frac{\tau_2}{t}\right)^{\mathcal{p}+\mathcal{q}+1}}{1-\frac{\tau_2}{t}}\right\}$
$$= \frac{1}{(\tau_1-\tau_2)\,t^{\mathcal{p}+\mathcal{q}}}\left(\frac{t^{\mathcal{p}+\mathcal{q}+1} - \tau_1{}^{\mathcal{p}+\mathcal{q}+1}}{t-\tau_1} - \frac{t^{\mathcal{p}+\mathcal{q}+1} - \tau_2{}^{\mathcal{p}+\mathcal{q}+1}}{t-\tau_2}\right)$$
$$= \frac{1}{(\tau_1-\tau_2)^{\mathcal{p}+\mathcal{q}}}\left\{\frac{t^{\mathcal{p}+\mathcal{q}+1}(\tau_1-\tau_2) - t(\tau_1{}^{\mathcal{p}+\mathcal{q}+1} - \tau_2{}^{\mathcal{p}+\mathcal{q}+1}) + (\tau_1{}^{\mathcal{p}+\mathcal{q}} - \tau_2{}^{\mathcal{p}+\mathcal{q}})}{(t-\tau_1)(t-\tau_2)}\right\}$$
$$= \frac{t}{t^{\mathcal{p}+\mathcal{q}}(t^2-3t+1)}\left(t^{\mathcal{p}+\mathcal{q}+1} - t\mathcal{T}_{\mathcal{p}+\mathcal{q}+1} + \mathcal{T}_{\mathcal{p}+\mathcal{q}}\right)$$
This completes the proof.

Corollary 3.15.1: For $(\mathcal{p}+\mathcal{q}) \to \infty$, we get
$$\sum_{\mathcal{p}+\mathcal{q}=0}^{\infty}\mathcal{T}_{\mathcal{p}+\mathcal{q}}t^{-(\mathcal{p}+\mathcal{q})} = \frac{t}{(t^2-3t+1)} \tag{3.21}$$

Theorem 3.16: (Explicit formula) For $n \geq 1$,
$$\mathcal{T}_n = \sum_{i=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}\binom{n-i-1}{i}3^{n-2i-1}(-1)^i \tag{3.22}$$
Proof: The proof is clear from the generating function (3.19).

3.3 Matrix Representation of Product of Fibonacci and Lucas Numbers

In this section, we present two cross two matrices for product of Fibonacci and Lucas numbers are given by

$$M = \begin{bmatrix} 3 & 1 \\ -1 & 0 \end{bmatrix}.$$

Theorem 3.17: For $n \in \mathbb{Z}$, we have
$$\begin{bmatrix} \mathcal{T}_{n+1} \\ -\mathcal{T}_n \end{bmatrix} = M\begin{bmatrix} \mathcal{T}_n \\ -\mathcal{T}_{n-1} \end{bmatrix} \tag{3.23}$$
Proof: To prove the result, we will use induction on *n*.
(3.23) is true for *n* = 1.
Suppose (3.23) is true for *n*, we get

$$\begin{bmatrix} \mathcal{T}_{n+2} \\ -\mathcal{T}_{n+1} \end{bmatrix} = \begin{bmatrix} 3\mathcal{T}_{n+1} - \mathcal{T}_n \\ -\mathcal{T}_{n+1} \end{bmatrix}$$
$$= \begin{bmatrix} 3 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \mathcal{T}_{n+1} \\ -\mathcal{T}_n \end{bmatrix}$$
$$= \begin{bmatrix} 3 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \mathcal{T}_n \\ -\mathcal{T}_{n-1} \end{bmatrix}$$
$$= \begin{bmatrix} 3 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 3\mathcal{T}_n - \mathcal{T}_{n-1} \\ -\mathcal{T}_n \end{bmatrix}$$
$$= \begin{bmatrix} 3 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \mathcal{T}_{n+1} \\ -\mathcal{T}_n \end{bmatrix}$$
$$= M \begin{bmatrix} \mathcal{T}_{n+1} \\ -\mathcal{T}_n \end{bmatrix}$$

This completes the proof.

Theorem 3.18: For $\boldsymbol{n} \in \mathbb{Z}$, we have

$$\begin{bmatrix} \boldsymbol{\mathcal{T}_{n+1}} \\ -\boldsymbol{\mathcal{T}_n} \end{bmatrix} = \boldsymbol{M^n} \begin{bmatrix} \boldsymbol{\mathcal{T}_1} \\ -\boldsymbol{\mathcal{T}_0} \end{bmatrix} \tag{3.24}$$

4. CONCLUSION

This study presents the properties of product of Fibonacci and Lucas numbers with the help of their Binet's formula and generating function. The concept can be executed for generalized second order sequences as well as polynomials. Also, present its two cross two matrices and find exciting properties such as the nth power of the matrix. Sequence of product of Fibonacci and Lucas numbers can also be called the sequence of alternate Fibonacci numbers. The details of which are in following table 2:

Table 2. Sequence of alternate Fibonacci numbers

$\boldsymbol{n}$	0	1	2	3	4	5	6	7	8	…
$\boldsymbol{F_n}$	0	1	1	2	3	5	8	13	21	…
$\boldsymbol{\mathcal{T}_n}$	**0**		**1**		**3**		**8**		**21**	**…**

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