Non Split Geo Chromatic Number of Certain Classes of Graphs

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ABSTRACT
In this paper, a new concept, Non – Split Geo chromatic number, which is a combination of non split geodetic set and chromatic set and is denoted by $g_{cns}(G)$ is introduced. Here its verified under what conditions $g_{cns}(G) = g_{ns}(G)$ for various classes of graphs and if they are equal, greater or lesser by comparing them to another graph concept viz., vertex covering number or edge covering number. We also determine few bounds with respect to $\alpha_0(G)$ and $\beta(G)$.

General Terms
Primary 05C12, Secondary 05C38, 05C05

Keywords
Distance, chromatic, geo chromatic, non split, vertex – cover, edge- cover, geodetic set, geodetic number, corona, cartesian products.

1. INTRODUCTION
The user may refer to [3], [9] and [10] for basic terminologies and definitions. Unless mentioned all graphs considered here are simple, loop-less and undirected. The concept of geodesics was first introduced by Gary Chartrand et al in [5]. Let G be a graph. If u, v $\in$ V(G), then u – v geodesic of G is a shortest path between u and v. The closed interval I[u, v] consists of all vertices lying in same u –v geodesic of G. For S $\subseteq$ V(G) the set I[S] is the union of all set of G if I[u, v] for all u, v $\in$ S. A set S is a geodetic set of G if I[S] = V(G). More about the concept of intervals can be studied in the following articles, [11], [12],[13]. The closed intervals I[u, v] in a connected graph G were studied and characterized by Nebesk´y [12], [13] and were also investigated extensively in the book by Mulder [11], where it was shown that these sets provide an important tool for studying metric properties of connected graphs. The intervals of an oriented graph have been studied in [7]. The cardinality of a minimum geodetic set of G is the geodetic number of G, denoted by g(G), studies related to this can be studied in [1], [4], [5], [6], [7],[8]. A set S $\subseteq$ V(G) is a non – split geodetic set in G if S is a geodetic set and <V(G)-S> is connected [15]. A set C $\subseteq$ V(G) is called a chromatic set, if C contains all p vertices of different colors in G, the minimum cardinality among all chromatic sets is called chromatic number denoted by $\chi(G)$. A geo chromatic set number S_c $\subseteq$ V(G) is both a geodetic and a chromatic set [14]. In this paper, a new concept is introduced, namely, Non – Split Geo chromatic number, which is a combination of non split geodetic set and chromatic set and is denoted by $g_{cns}(G)$. In the next section, for some standard class of graphs, its verified $g_{cns}(G) = g_{ns}(G)$ and under what conditions they are equal, greater, lesser to vertex covering number or edge covering number. Also few bounds with respect to $\alpha_0(G)$ and $\beta(G)$ are determined

2. PRELIMINARY RESULTS
1. Every geodetic set of a graph contains the extreme vertices.
2. For any tree T with d pendant vertices g(T) =k.
3. For any non tree graph G of order n, $\alpha_1(G) + \beta_1(G) = n$.
4. For cycle C_n of order n $\geq$ 3, $g(C_n)=\begin{cases}2, \text{ if n is even}\\ 3, \text{ if n is odd.}\end{cases}$
5. For any graph G, g(G) $\leq$ $g_{ns}(G)$.
6. For any cycle C_n of order n ≥ 3, $g_{ns}(C_n)=\begin{cases}\frac{n}{2}+1, \text{ if } n \text{ is even}\\ \left\lfloor\frac{n}{2}\right\rfloor+2, \text{ if } n \text{ is odd}.\end{cases}$

Before proceeding, with the result, the definition of non split geo chromatic number of graph is been introduced below.
The non split geo chromatic number $g_{cns}(G)$ of a connected graph G is the minimum cardinality of a non split geodetic set combined along with the [minimum usage of colors to have a proper coloring of the G] minimum color classes used.

3. RESULTS ON NON SPLIT GEO CHROMATIC NUMBER ON VARIOUS CLASSES OF GRAPHS
Theorem 3.1 **For a graph, G ≅ C_n of order n $\geq$ 3,** $g_{cns}(C_n)=\begin{cases}\frac{n}{2}+1, \text{ if } n \text{ is even}\\ \left\lfloor\frac{n}{2}\right\rfloor+1, \text{ if } n \text{ is odd}.\end{cases}$

Proof: Let G ≅ C_n be a cycle on n $\geq$ 3 vertices, for a cycle on n vertices, say C_{2p} = {$v_1,v_2,v_3,\ldots,v_{2p},v_1$}, the v_{p+1} is the antipodal vertex. Suppose S = {v_1, v_{p+1}} is the geodetic set and <V(G) - S > is not connected. Thus, S is not a non split geodetic set but S' = {$v_1,v_2,v_3,\ldots,v_{2p+1}$} is a non split geodetic set, hence $g_{ns}(G)$ $\leq$ p+1. If $S_1 \in$ V(G) [S_1 is any set] with |S| <| S' |, then S_1 contains at most p – elements, where $V'(G) - S_1$ is not contained. Therefore, the idea would be adding one more vertex. Thus, $g_{cns}(G)$ = p+1 = $\frac{n}{2}$ +1.
Continuing, the set considered above, i.e., S = (v_1, v_{p+1}) covers the color classes used , but since <V(G) - S > does not form a connected graph, upon its removal, adding one more vertex ie.,

S = {v_1, v_{p+1}, v_{p+2}}, thus results this set to form a non – split geodetic set along with covering the color classes used.

Case 1. For C_n, n being even, its known that g_{ns}(G) = p+1= $\frac{n}{2}$ +1. The total number of colors used for proper coloring of C_n = $\chi(C_n)$ = 2 {for n being even}. By looking at the geodetic set which is just S = {v_1, v_{p+1}} though this set covers up the color classes used, this does not satisfy to be a non – split geodetic set. Hence, adding one more vertex to the geodetic set, satisfies the required condition. Thus g_{cns}(G) = $\frac{n}{2}$ +1.

Case 2. For C_n, n being odd, its known that g_{ns}(G) = p+2= $\left\lfloor \frac{n}{2} \right\rfloor + 2$. Looking for the color class, $\left\lfloor \frac{n}{2} \right\rfloor$ covers up all the colors used, but the set {v_1, v_{p+1}, v_{p+2}} does not suffice to be the non– split geo chromatic set, hence, the set { v_1, v_{p+1}, v_{p+2}}+1, will be the g_{cns}(G), resulting in $\left\lfloor \frac{n}{2} \right\rfloor + 1$.

Theorem 3.2 For cycle G ≅ P_n of order n $\geq$ 2, $g_{cns}(P_n) = \begin{cases} 2, \text{ if } n \text{ is even} \\ 3, \text{ if } n \text{ is odd}. \end{cases}$

Proof: Let P_n be a path on n $\geq$ 2 vertices. The proof can be drawn for 2 cases, n being odd and even.

Case 1. Let P_n = {v_1,v_2,v_3,…,v_{2p}}, n being even for a path on even vertices, $\chi(P_n)$ = 2. The geodetic set is S = {v_1, v_{2p}}. The non – split geodetic number $g_{ns}(P_n)$ = 2. For this set to be containing the color classes used, these vertices suffices for the same set to satisfy. Hence, the set S = { v_1, v_{2p} } = g_{cns}(G)=2.

Case 2. For a path on odd vertices, $\chi(P_n)$ = 2. The geodetic set for a path on odd vertices S = {v_1, v_{2p+1}}. the non – split geodetic set S' = {v_1, v_{2p}, v_{2p+1}} and the same will suffice to be the non spilt geo chromatic set, since for an odd path, the end vertices do receive the same color, adding a neighborhood vertex of either v_1 or v_{2p+1}, covers up the color class and hence g_{cns}(G)=3.

Corollary For a star $K_{1,n}$, there exist no $g_{cns}(K_{1,n})$. Since the geodetic set covers up all the pendant vertices and the all these receive the same color, deleting them, will not result in a disconnected graph, but a K_1.

Theorem 3.3 For a tree on n vertices, g_{cns}(T) = k+1.

Proof: Let G be a acyclic connected graph with n vertices, out of which there are k pendants. Now, the g(G)=k, g_{ns}(G)=k. Also, now to find the cardinality of a non split geo chromatic set, the pendant receives a unique color and hence removal of all the k – pendants will still make the graph to be connected one. But deleting one more vertex along with k, not only results in a connected graph, but also satisfies the color classes used. Hence g_{cns}(T) = k+1.

Corollary For a path on 4 vertices, ie., P_4, k is the number of pendants, k $\leq \chi(P_4)$.

Theorem 3.4 For $K_{m,n}$, the $g_{cns}(K_{m,n}) = g_{ns}(K_{m,n})$ = m + n – 1.

Proof: Let G = $K_{m,n}$, where U = {u_1,u_2,u_3,…,u_m} and V = {v_1,v_2,v_3,…,v_n} be the partite set of G and m $\leq$ n, W = U $\cup$ V. It is shown that in [7], for $K_{m,n}$, g_{ns}(G) = m + n – 1. Now its inspected in terms of the chromatic class being a non – split geodetic number.

Case 1. Let K = U $\cup$ V – x, for x $\in$ V, for v_n $\in$ V, $1 \leq m \leq n-1$ lying on u_i – v_j geodesic for $1 \leq i \leq j \leq m$, such that S forms the geodetic set of G. This set S forms a non – split geodetic set G, since <V(G) – K> is connected covering up both color classes.

Case 2. Let $|K'| < |K|$. If K' is not a subset of U, then the induced graph m, V(G) – S' is disconnected, and so K' does not form a non – split geodetic set of G, through this covers up both the color classes, it fails to form non – split geodetic set.

Case 3. Similarly if K' = U, then K' is geodetic but not the subgraph induced from V(G) – K' is not connected and also this contains only one of the color classes. Thus K' does not form a non – split geo chromatic set.

Case 4: if K' = V - y, then K' is not a non – split geo chromatic set of G. Thus from the above, cases we can say $g_{cns}(K_{m,n})$ = $|K|$ = m + n – 1.

Note:

1) It is shown that g_{ns}(G) = n – 1 for G ≅ $K_{1,n}$.

2) It is shown in [15] that gns (G), where G ≅ K_n does not exist.

From the above two findings, the following results are true.

Corollary: g_{ns} (G) = n – 1 where G ≅ $K_{1,n}$ does not exist.

Corollary: g_{cns} (G), where G ≅ K_n, does not exist as it result in a null graph.

Corollary: g_{cns} ($W_{1,n}$) , does not exist. □

4. NON-SPLIT GEO-CHROMATIC NUMBER - GCNS, FOR CERTAIN GRAPH OPERATIONS

□

(i) Corona Graph: Let G and H be 2 graphs of order m. The corona product GoH is defined as the graph obtained from G and H, by taking one copy of G and m copies of H and joining by an edge, all the vertices from i^{th} copy of H with the i^{th} copy of G.

Theorem 4.1 Let G ≅ Cn be a connected graph of order n and H ≅ K_2, then g_{cns}(GoH) = 2n > $\alpha_0(G)$.

Proof: Let S denote the non – split set which is also the non – split geo chromatic, where each v_iv_j ϵV(H), for i=1,j=2. For each v_iv_j of all n copies, forms the non – split geodetic set along with covering all the color classes which is almost 3, for any C_n, n being odd or even. For g_{cns}(GoH), taking all copies of K_2 in accordance with the $|n|$ of the cycle, given the value being equal to 2n. Hence, g_{cns}(GoH) = 2n

(ii) Adding the pendant edge

An uv edge becomes a pendant edge, when deg (u) =1 and deg(v)>1, and u is called the pendant vertex .

Theorem4.2 Let G be a graph formed by adding n K_2's on each of the pendant vertices of a P_5, then, $g_{cns}(G_1) = g(G)$.

Proof: Let S denotes the set of all pendants, S= $\{u_1,u_2,u_3,..,u_n,v_1,v_2,v_3,\ldots,v_n\}$. Deletion of these pendants makes the graph disconnected, but the color used for all these pendants will be same. Hence, deletion of any of the pendant vertices from the path which uses the 2nd color, gives $g_{cns}(G) = g(G)$.

Theorem 4.3 If $G \cong P_n$, then, let GoK2 = G'. $g_{cns}(G') = gns(G) = \alpha_0(G)$. = n.$K_2$.

Proof: Let $v_1, v_2, v_3,\ldots,v_n$ be the vertices of the path then (u_1,u_2), (u_3,u_4) be the K_2's that goes for the corona product with P_n. Then, the set of K_2's added forms the non split geodetic set. Let the color classes used for the path consists $\{\lambda_1, \lambda_2\}$ irrespective of path length. Similarly for the K_2's, that are added up, forms a cycle on three vertices, hence we require minimum three colors, when the corona operation is performed. Thus, using the alternative color for the K_2's covers all the color classes and also forms a non – split geo chromatic set. Thus, the set is also the vertex covering number of this graph. Hence $g_{cns}(G') = g_{ns}(G) = \alpha_0(G)$.

Theorem 4.4 Let $G \cong C_n$ (for n > 3), then G' is formed by adding a, pendant edge to any one of the vertices of G. Then $g_{cns}(G') = g_{ns}(G) \geq \alpha_0(G)$ and

$g_{cns}(G') = \begin{cases} 2, \text{if } n \text{ is even} \\ 3, \text{if } n \text{ is odd} \end{cases}$.

Proof: Let $\{v_1, v_2, v_3,\ldots,v_n, v_1\}$be a cycle with $|C_n|$ = n. Let G' be a graph that is obtained from $G \cong$ Cn by adding a pendant edge (v_i, u), for $1 \leq i \leq n$ such that $v_i \in C_n \cong G$ and $v_i \notin C_n \cong G$. we consider 2 cases, one for odd length and one for even length.

Case 1. For $G \cong C_{2n}$, let $S = \{v_i, u\}$be the non – split geodetic set of G, where U is the pendant vertex of G' and the diam(G') = $\{v_i - u\}$ path. Clearly $I[S] = V[G']$. Also for all w,x $\in V(G') - S$, <V(G') – S> is connected. For the coloring of this $G', \chi(G) = 2,$ such that if the vertex u, receives color λ_1 , the vertex adjacent of u, viz v_i receives color λ_2 , and for the $\{v_i - u\}$diametrical path receives the alternate color. The sub graph <V(G') – S> still remains to be connected and hence $g_{cns}(G') = 2$ for n being even.

Case 2: Similarly, when $G \cong C_{2n+1}$, then , upon adding a new vertex, the number of vertices to cover the color classes along with being a non split geodetic set , is 3.

Corollary: Let $G \cong$ C3, then adding pendant vertices to any one of the vertices results in $g_{cns}(G) \neq g_{ns}(G)$. The vertices if C_3 are u_1, u_2 and u_3. Let u_4 be the new vertex that is added. As n(G) = 3, g(G) = $\{u_1, u_4\}$. $g_{ns}(G)$= $\{u_4\}$. Hence they are not equal.

5. CORONA PRODUCT OF NON SPLIT GEO CHROMATIC NUMBER OF GRAPH

Theorem 5.1 Let G be a graph obtained from the corona product of P_n with P_2. Then, $g_{cns}(G) = g_{cns}(P_n \circ P_2) = n.P_2$

Proof: Let G represent $P_n \circ P_2$. Then for any path $\chi(P_2)$ = 2. Also the $g(P_n) = 2$, the set of pendant vertices. By taking the corona product, we have $n(P_2)$ which is the geodetic set of the graph G. The graph, when colored requires a minimum of 3 colors to properly color the graph. Hence, the geodetic set upon deletion: which also is the non – split set retains the graph to be still connected set,. Hence $g_{cns}(P_n \circ P_2) = n.P_2$

Corollary: Let G denote $G = C_n \circ P_2$, then $g_{cns}(G) = n.P_2$

Proof: In the product of C_n with P_n , the $\chi(G) = 3$. Then g(G) is the set of all P_2's . The non – split geodetic set is again the set of all the P_2, as their removal, results in a non – disconnected graph. Now for the non – split geo chromatic number, the K_2's that are removed covers up all the color classes, hence, $g_{cns}(G) = n.P_2$.

Corollary: There exists no $g_{cns}(C_n \circ P_2)$ for n being even.

Proof: Since, the vertices that form the geodetic set □es the same color class and deletion of one more vertex will not form the geodetic set.

6. CARTESIAN PRODUCT OF GRAPHS

The Cartesian product, G □ H of graphs G and H is a graph such that, the vertex set of G □ H is the Cartesian product V(G) × V(H) and two vertices (u,u') and (v,v') are adjacent in G □ H if and only if either u = v and u' is adjacent to v' in H, or u' = v' and u is adjacent to v in G.

Theorem 6.1 For a graph $G = P_n \square P_2$, the non – split geo chromatic number is always 2, for n being odd.

Proof: The graph formed by taking the Cartesian product of a path P_n with P_2 has diameter n, and hence the initial vertex and the final vertex along the diametrical path forms the geodetic set and to color this product graph atmost 2 colors are required and by deleting theses extreme vertices that lie on the diametrical path, the graph still remains to be connected. Hence $g_{cns}(G)=2$.

Corollary: There exist no $g_{ncs}(P_n \times P_2)$ for n being even, ie., for n > 1.

Since the geodesic consists of the initial vertex and its eccentric vertex where both belong to same color class and deleting this retains the graph to be still connected, where the other color class is not being used in the geodetics set.

Theorem 6.2 For any $G = C_n \times P_2$, for n being odd,

$g_{ncs}(G)= 3$.

Proof: Let the graph C_n being properly colored, then n be any vertex in C_n colored with color λ_1, then eccentric vertex that lies in the other cycle. These vertices are colored with the other 2 remaining colors say λ_2, λ_3. Hence deletion of these vertices that forms the geodetic set, upon deletion retains the graph to be still connected. Hence the number equals the color classes ($\lambda_1, \lambda_2, \lambda_3$) used to properly color the product graph. Thus $g_{ncs}(G)= 3$.

Corollary: There exist no $g_{cns}(C_n \times P_2)$ for n being even.

Proof: The vertices that form the geodetic set uses the same color class and deletion of one color vertex will not form the geodetic set.

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8. CONFLICT OF INTEREST

I confirm that both the authors of the manuscript have no conflict of interests to declare. I confirm that both authors listed on the title page have contributed significantly to the work of this article.

9. CONCLUSION/FURTHER SCOPE

This concept can be further extended to the other class of graphs and can be verified for which class of graphs the $g_{cns} = g_{ns} = \chi(G)$.

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