# Credit Financing Scheme for Imperfect Quality Items with Allowable Shortage and Learning Effects on Screening Process

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## ABSTRACT
The advent of new technologies, systems, trend workforce and new applications in manufacturing/production sector has undoubtedly lightened workloads. However occasional diversifications in production system cannot be completely eliminated. Each produced/ordered batch may contain a fraction of defective items which can vary from batch to batch. Nevertheless, defective items are often removed from high quality batch through a discrete screening procedure. Thus, a screening process is considered an essential task in technology -based industries, with the sole objective of ensuring customer satisfaction. Furthermore, the repetition of same tasks enhances workers efficiency. Additionally, credit financing has been recognized as an impressive promotional tool to attract new customers and serve as an effective incentive scheme for retailers.

Based on this scenario, the present article proposed an inventory model for a retailer dealing with imperfect quality items under permissible delays in payment. A screening process was applied to each batch to separate good and defective items and learning effects were analyzed with allowable shortages under fully backlogged demand. This model developed such strategies so that the order quantity, shortages and the number of repetitions on screening are optimized and the expected total profit may be maximum. A mathematical model was formulated to represent this scenario. The results were validated using numerical examples and a comprehensive sensitivity analysis was conducted with respect to key parameters.

**Keywords**
Imperfect quality items Screening, Shortage Credit financing, Learning effects.

AMS Subject Classification: 90B05, 90B30, 90B50

## 1. INTRODUCTION

The traditional parsimonious order quantity (EOQ) models, although functional but often impractical because that are based on restrictive and fanciful assumptions, due to which they inconvenient to use for many industries today. In the real life scenario, the inventory model should include certain characteristics which demonstrate the real inventory circumstance. Out of many real life factors, imperfect production is one such factor which can not be neglected, because in the inventory system very often a mixer of both good and defective items may include and the inventory value is dependent on the product quality and customers satisfaction. Very often imperfect quality may be considered as spoilage, damage, decay, obsolescence, evaporation, etc. consequently, the usefulness of the original one is affected. In many situations for items such as steel, hardware, glassware and toys in which for smal batch size, the rate of defective items is low in each case, there is slight need to consider imperfect quality factor for obtaining of the economic batch size. However, in the large scale industries batch size can not be small, in such cases the rate of defective items can not be neglected.

Singa et al. [16] proposed an EPQ model for imperfect quality items, where the production rate of defective items was governed by an expected random variable. They assumed that the defective items were separated through a screening process. Chiu [2] developed an EPQ model that incorporated the reworking of defective items into their economic production model, allowing for backlogging. This approach differed from the classic EPQ model, which assumes a perfectly functioning manufacturing facility. Chung K. J. and Hou K. L. [3] formulated a cost function by optimizing run time for a deteriorating production system under allowable shortage. In this system, it was evident that the elapsed time until the production process shift was arbitrarily distributed during a production run. Daya and Hariga [1] studied the effects of imperfect production procedure on economic lot scheduling problem (ELSP). It has been assumed that after system gets deterioration of facility with time and shifts at a random time to an out of control state and begins to produce defective items..

The traditional EPQ/EOQ model was extended by Salameh and Jaber [13] through incorporating of an imperfect quality items into the EPQ/EOQ formulation. It was assumed in the study that poor-quality items were sold as a single batch at the end of the screening process. Hayek and Salameh [6] investigated the effects of imperfect quality items on the finite production system incorporating a rework process for defective items at a constant rate following production stoppage. In this study the percentage of defective items was treated as a random variable, characterized by a known probability density function. Sana [12] investigated the random time in which the production facility transitioned from an in-control state to an out-of-control state for an imperfect production system. In this article they removed the barrier of the basic assumption of the classical EPL model is that 100% of produced items are perfect quality which not valid for most of the production environments.

A single-stage manufacturing system of an economic production quantity (EPQ) under planned backorders model with rework process was developed by Sarkar *et el*. [14]. A production model with possible preventive measures to reduce imperfect production was developed by Shah et al. [15]. This model was designed on the basis of stocking strategies screening and maintain customer goodwill. Production models based on two categories of defective products-reworkable and rejected items- were proposed by Kang et al. [7]. Four types of distribution density functions, including uniform, triangular, double triangular and beta distribution were used to represent the defective items produced.

The effects of learning on economical lot-sizing models were investigated by Marchi et al. [10] and strategies to optimize cost functions associated with imperfect quality items were formulated.

The effect of imperfect quality items on a production system was analyzed by Khanna et al. [8], and has been further developed under the inspection and rework process. Two situations for the imperfect quality items have been handled: either they are sold at a reduced price or reworked. A two stage production model was designed by Glock and Jaber [5], incorporating the effects of learning and forgetting under the position of a potential bottleneck in the system, where the first stage is used to make semi-finished items and second stage is used to make finished items. An inventory model was constructed by Gautam et al. [6] to handle only defective products, applying a proficient rework to make the product fit to be sold at the primary price. The demand for the product has been assumed to depend on the selling price and advertisements.

Every produced or ordered lot may contain some fraction of defectives, which may differ from process to process. The situation becomes more susceptible when the items are deteriorating in nature. A credit financing policy for deteriorating imperfect quality items was constructed by Khanna et al. [9], incorporating allowable shortages, in which a screening procedure was also applied to separate the good and defective items. In this article an EOQ model for imperfect quality items under credit financing scheme allowing with shortage has been developed. Furthermore, the learning effects on the screening process have been analyzed under the learning curve effects. We have used Sigmoid function (curve) which is governed by the formula $x(n) = \left(\frac{A}{G+e^{Bn}}\right)$, where $x(n)$ is defective percentage rate of item in the single batch, n number of efforts, $A, B, G > 0$ are constants parameters as Wright T. P.[17] and Nigwal et al [11].

## 2. NOTATIONS AND ASSUMPTIONS

### 2.1 Assumptions: Assumptions that are used to develop the model

- The demand rate is deterministic constant and continuous,
- Shortages are allowed, Backlogged demand is fulfilled it right time up after completion of screening process,
- The supplier offers a certain credit period M to settle the account to the retailer,
- The screening process and demand proceeds simultaneously, but $\boldsymbol{\lambda > D}$,
- An interest rate is charged by supplier between the interval $\boldsymbol{M}$ to $\boldsymbol{T}$.
- The credit period $\boldsymbol{M}$ lies between $\mathbf{0}$ to $\boldsymbol{T}$.

### 2.2 **Notations**: Notations that are used to develop the model

1. $\boldsymbol{y}$: Ordering cost,
2. $\boldsymbol{c_p}$: chasing cost per unit for retailer,
3. $\boldsymbol{h_c}$: Holding cost per unit item per unit time,
4. $\boldsymbol{x(n)}$: Fraction of imperfect quality items,
5. $\boldsymbol{\lambda}$: Screening rate in units/unit time,
6. $\boldsymbol{\beta}$: Screening cost per unit items,
7. $\boldsymbol{t_s}$: Screening time, where $\boldsymbol{t_s = \frac{\phi_n}{\lambda}}$,
8. $\boldsymbol{\phi_n}$: Ordered quantity,
9. $\boldsymbol{T}$: Replenishment cycle length,
10. $\boldsymbol{D}$: Constant demand rate/unit time,
11. $\boldsymbol{I_e}$: Earned interest per unit time,
12. $\boldsymbol{I_p}$: Paid interest per unit time,
13. $\boldsymbol{b}$: Allowable maximum backordered level per cycle,
14. $\boldsymbol{p}$: Selling price per unit item,
15. $\boldsymbol{M}$: Permissible delay period in square off the account,
16. $\boldsymbol{c_1}$: Shortage cost per unit per unit time,
17. $\boldsymbol{c_s}$: Selling price per unit of imperfect quality items,
18. $\boldsymbol{I_1(t)}$: Inventory level during the time interval $\boldsymbol{(0, t_s)}$,
19. $\boldsymbol{I_2(t)}$: Inventory level during the time interval $\boldsymbol{(t_s\ \ t_1)}$.

## 3. THE MATHEMATICAL MODEL

The problem of initial batch size $\boldsymbol{\emptyset_n}$ supplied instantaneously to a retailer with the purchasing price $\boldsymbol{c_p}$ per unit and ordering cost $\boldsymbol{y}$ has been encounterd. As per assumptions each delivered batch contains some fractions of imperfect quality items $\mathbf{x(n)}$. A 100% screening process is applied on received upon each received batch at a screening rate $\boldsymbol{\lambda}$ per unit time. The imperfect quality items are separated through screening and kept in stock as single batch and sold at a discounted price $\boldsymbol{C_s}$ per unit at the end of the cycle where $\boldsymbol{c_p > C_s}$. Let D be the demand rate per unit time, $\boldsymbol{t_s}$ be the screening time for unit cycle and $\boldsymbol{T}$ be the cycle length. It has been assumed that the backlogged demand is fulfilled after the completiting screening. Moreover, let b be the backordered level, and t2 the time to build backordered level of b units at the rate D, $\boldsymbol{t_2 = \frac{b}{D}}$. During the time interval$\boldsymbol{(0, t_s)}$ and $\boldsymbol{(t_s, t_1)}$ inventory levels be $\boldsymbol{I_1(t)}$ and $\boldsymbol{I_2(t)}$, respectively. Due to demand inventory level $\boldsymbol{I_2(t)}$ reaches zero at $\boldsymbol{t_1}$. At the time $\boldsymbol{t_1}$ the backordered demand starts building at the demand rate $\boldsymbol{D}$ until the initiating of a new cycle, when a new batch of size $\boldsymbol{\emptyset_n}$ is received. The backordered level is still pending shortly after receiving the new batch until screening process of the new batch is completed at the time $\boldsymbol{t_s}$. After instantaneously backordered demand is fulfilled. Let $\boldsymbol{I_1(t)}$ be the inventory level in the time interval $\boldsymbol{(0, t_s)}$, at any movement inventory status is represented by the differential equation

$$\boldsymbol{\frac{dI_1(t)}{dt} = -D, \qquad 0 \le t \le t_s \qquad (3.1)}$$

With the initial and boundary conditions at $\boldsymbol{t = 0}$, $\boldsymbol{I_1(t) = \emptyset_n}$ and at $\boldsymbol{t = t_s}$, $\boldsymbol{I_1(t_s) = (1 - x(n))\emptyset_n}$

The solution of the equation (3.1) together with the boundary condition $\boldsymbol{t = t_s}$, $\boldsymbol{I_1(t_s) = (1 - x(n))\emptyset_n}$ is

$$\boldsymbol{I_1(t) = -Dt + Dt_s + (1 - x(n))\emptyset_n \qquad (3.2)}$$

Since the screening process is applied on the batch of imperfect quality items, therefore after the screening process the number of defective items at time $t_s$ will be $x(n)\emptyset_n$ and let the back ordered will be b. Furthermore the effective inventory level at $t = t_s$, after removing the defective items and backorders is given by

$$I_{eff}(t_s) = (1 - x(n))\emptyset_n - b \qquad (3.3)$$

Let $I_2(t)$ be the another inventory level at any time interval $(t_s \le t \le t_1)$.

The inventory status for the period $t_s \le t \le t_1$ is governed by

$$\frac{dI_2(t)}{dt} = -D, \qquad t_s \le t \le t_1 \qquad (3.4)$$

With the initial and boundary conditions at $t = t_s$, $I_2(t) = I_{eff}(t_s)$.

The solution of the equation (3.4) together with the boundary condition $t = t_s$, $I_2(t) = I_{eff}(t_s)$ is given by

$$I_2(t) = D(t_s - t) + (1 - x(n))\emptyset_n - b \qquad (3.5)$$

We have to solve the equation for $t_1$ with the condition $t = t_1, I_2(t_1) = 0$.

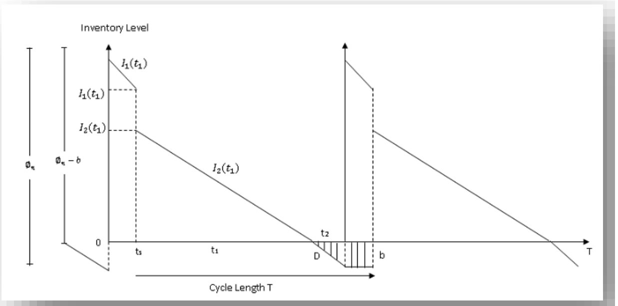


*Figure-1*

$$t_1 = \frac{Dt_s + (1 - x(n))\emptyset_n - b}{D} \qquad (3.6)$$

$$T = t_1 + t_2 \qquad (3.7)$$

Where $t_1 = \frac{b}{D}$

The present article has been developed on the basis of permissible delay in payment. therefore, retailer's total profit function $z_j\ j = 1, 2, 3\ viz.$ must be depended upon the credit period, We focus on three distinct possible cases, which was explained here

*3.1 Case 1:* $0 \le M \le t_s$
*3.2 Case 2:* $t_s \le M \le t_1$
*3.3 Case 3:* $t_1 \le M \le t_2$

retailers total profit function can be determined by considering the following components:

$$z_j = Sales\ revenue - Ordering\ cast - screening\ cost - Holding\ cost - shortage\ cost + Earned\ interest - Paid\ interest \qquad (3.8)$$

1. Total Sales revenue = sales of perfect quality items + sales of imperfect quality items $= p\big(1 - x(n)\big)\emptyset_n + C_s\, x(n)\emptyset_n$ (3.9)
2. Ordering cost per cycle = $y$ (3.10)
3. Screening cost per unit items $= \beta\emptyset_n$ (3.11)
4. Shortage cost $= c_1 b(t_2 + 2t_s)$ (3.12)
5. Holding cost during the two time interval $[0,\ t_s]$ and $[t_s, t_1]$

$$= h_c\left[\int_0^{t_s} I_1(t)dt + \int_{t_s}^{t_1} I_2(t)dt\right]$$

$$= h_c\left[Dt_1t_s - \frac{Dt_1^2}{2} + (1 - x(n))\emptyset_n t_1 + b(t_s - t_1)\right] \qquad (3.13)$$

To get the formula of last two components viz earned and paid interest were considered the following three different cases

Case 1:$0 \le M \le t_s$

In this case retailer can earn interest on revenue generated from the sales up to credit period $M$. After the credit period $M$ retailer can pay the interest at some specified rate from $M$ to $T$.

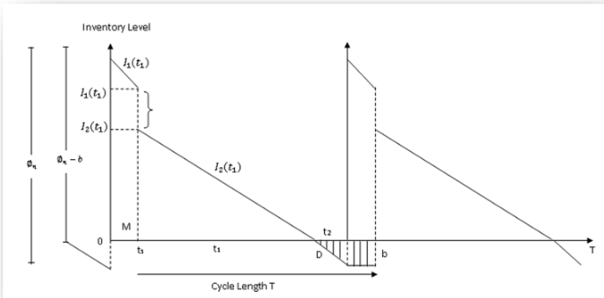


Figure-2

$$\text{Interest Earned } = pI_e\int_0^{t_s} Dtdt = pI_e\frac{DM^2}{2} \qquad (3.14)$$

$$\text{Interest Payable } = c_pI_p\left[\int_M^{t_s} I_1(t)dt + \int_{t_s}^{t_1} I_2(t)dt\right]$$

$$= c_pI_p\left[Dt_1\left(t_s - \frac{t_1}{2}\right) + (1 - x(n))\emptyset_n(t_1 - M) - DM\left(t_1 - \frac{M}{2}\right) + b(t_s - t_1)\right] \qquad (3.15)$$

On inserting the values of various components from equations $[(3.9) - (3.15)]$ in to the question (3.8) the total profit function for **Case 1, $z_1(\emptyset_n,\ b)$**becomes

$$z_1(\emptyset_n,\ b) = p\big(1 - x(n)\big)\emptyset_n + C_s\, x(n)\emptyset_n - y - \beta\emptyset_n - c_1b(t_2 + 2t_s) - h_c\left[Dt_1t_s - \frac{Dt_1^2}{2} + (1 - x(n))\emptyset_n t_1 + b(t_s - t_1)\right] + pI_e\frac{DM^2}{2} - c_pI_p\left[Dt_1\left(t_s - \frac{t_1}{2}\right) + (1 - x(n))\emptyset_n(t_1 - M) - DM\left(t_1 - \frac{M}{2}\right) + b(t_s - t_1)\right] \qquad (3.16)$$

**Proposition3.1:** *An order quantity $\boldsymbol{\phi_n}$ and credit period back ordered level $\boldsymbol{b}$ have an optimal point $(\boldsymbol{\phi_n^*}, \boldsymbol{b^*})$,*

***Proof:*** As per the conditions of optimality the total profit function has an optimal point $(\boldsymbol{\phi_n^*}, \boldsymbol{b^*})$, if the first order partial derivatives are disappears at point $(\boldsymbol{\phi_n^*}, \boldsymbol{b^*})$, , i.e. $\frac{\partial z_1(\emptyset_n, b)}{\partial \emptyset_n} = 0$ and $\frac{\partial z_1(\emptyset_n, b)}{\partial b} = 0$ Therefore,

$$p\big(1 - x(n)\big) + C_s\, x(n) - \beta - \frac{2c_1b}{\lambda} - h_c\left[\frac{Dt_1}{\lambda} - \frac{D\phi_n t_1'}{\lambda} - \right.$$

$Dt_1't_1 + (1 - x(n))\emptyset_n t_1' + (1 - x(n))t_1' + b(\frac{1}{\lambda} - t_1')\Big] - c_p I_p \Big[Dt_1\left(\frac{1}{\lambda} - \frac{t_1'}{2}\right) + Dt_1'\left(\frac{\phi_n}{\lambda} - \frac{t_1}{2}\right) + (1 - x(n))(t_1 - M) + (1 - x(n))\emptyset_n t_1' - DMt_1' + b(\frac{1}{\lambda} - t_1')\Big] = 0 \qquad (3.16a)$

$$\frac{2c_1 b}{D} - \frac{2c_1\emptyset_n}{\lambda} - h_c\left[\frac{b}{D} - \frac{(1 - x(n))\emptyset_n}{D}\right] - c_p I_p\left[\frac{b}{D} - \frac{(1 - x(n))\emptyset_n}{D} + M\right] = 0 \qquad (3.16b)$$

The optimal value of $\boldsymbol{\phi_n}$ **and** $\boldsymbol{b}$ can be obtained from the Solution of the above system of two equations.

**Proposition 3.2:** The total cost function $z_1(\emptyset_n,\ b)$ shows jointly concavity for an optimum point $(\boldsymbol{\phi_n}, \boldsymbol{b})$ if $RT - S^2 > 0$, and $R < 0$. i.e

$h_c\left[\frac{2Dt_1'}{\lambda} - Dt_1'^2 + 2(1 - x(n))t_1' + b(\frac{1}{\lambda} - t_1')\right] - c_p I_p\left[2Dt_1'\left(\frac{1}{\lambda} - \frac{t_1'}{2}\right) + 2(1 - x(n))t_1'\right]\left[\frac{2c_1}{D} + \frac{h_c}{D} + \frac{c_p I_p}{D}\right] - \left[\frac{2c_1}{\lambda} - c_p I_p\left[\frac{(1 - x(n))}{D} + t_1'\right]\right]^2 > 0$, and $h_c\left[\frac{2Dt_1'}{\lambda} - Dt_1'^2 + 2(1 - x(n))t_1' + b(\frac{1}{\lambda} - t_1')\right] c_p I_p\left[2Dt_1'\left(\frac{1}{\lambda} - \frac{t_1'}{2}\right) + 2(1 - x(n))t_1'\right] < 0. \qquad (3.16c)$

***Proof:*** *After differentiate twice partially of equation* (3.16) *with respect to* $\emptyset_n$, $\boldsymbol{b}$, can obtain the derivatives

$\frac{\partial^2 z_1(\emptyset_n,b)}{\partial\emptyset_n^2} = -h_c\left[\frac{2Dt_1'}{\lambda} - Dt_1'^2 + 2(1 - x(n))t_1' + b(\frac{1}{\lambda} - t_1')\right] - c_p I_p\left[2Dt_1'\left(\frac{1}{\lambda} - \frac{t_1'}{2}\right) + 2(1 - x(n))t_1'\right] \qquad (3.16d)$

$\frac{\partial^2 z_1(\emptyset_n,b)}{\partial b^2} = -\left[\frac{2c_1}{D} + \frac{h_c}{D} + \frac{c_p I_p}{D}\right] \qquad (3.16e)$

$\frac{\partial^2 z_1(\emptyset_n,b)}{\partial\emptyset_n b} = -\frac{2c_1}{\lambda} + c_p I_p\left[\frac{(1 - x(n))}{D} + t_1'\right] \qquad (3.16f)$

The Hassian matrix H of the profit function $\boldsymbol{z_1(\emptyset_n,\ b)}$ **is** square matrix of order 2 of partial derivatives of $\boldsymbol{z_1(\emptyset_n,\ b)}$**:**

$$\mathrm{H}(\emptyset_n,\ \boldsymbol{b}) = \begin{bmatrix} \frac{\partial^2 z_1(\emptyset_n,b)}{\partial\emptyset_n^2} & \frac{\partial^2 z_1(\emptyset_n,b)}{\partial\emptyset_n b} \\ \frac{\partial^2 z_1(\emptyset_n,b)}{\partial\emptyset_n b} & \frac{\partial^2 z_1(\emptyset_n,b)}{\partial b^2} \end{bmatrix}$$

The determinant of this Hassian matrix is positive and $\frac{\partial^2 z_1(\emptyset_n,b)}{\partial\emptyset_n b}$ is negative at an optimum point $(\boldsymbol{\phi_n^*}, \boldsymbol{b^*})$, , this condition has been satisfied and illustrated with numerical data analysis and examples. After simplification of above we can obtain above equation $(3.16c)$, This shows the jointly concavity property of profit function.

### *Case 2:* $\boldsymbol{t_s \le M \le t_1}$

In this case retailer can earn interest on revenue generated from the sales up to credit period $M$. Retailer can also earn interest for the shortage which yield during the period $M - t_s$ and due to the sale of defective items during $M - t_s$.

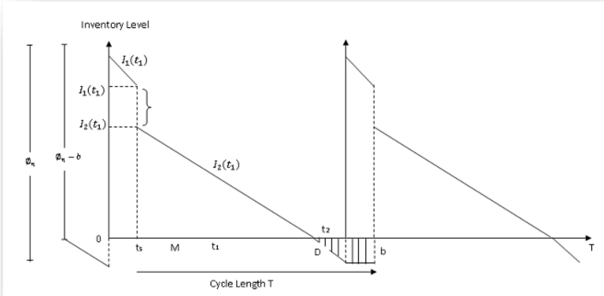


Figure-3

Interest Earned $= pI_e\left[\int_0^M Dt\,dt + b(M - t_s)\right] + C_s x(n)\emptyset_n I_e(M - t_s)$

$$= pI_e\left[\frac{DM^2}{2} + b(M - t_s)\right] + C_s x(n)\emptyset_n I_e(M - t_s) \qquad (3.17)$$

Interest Payable $= c_p I_p\left[\int_M^{t_1} I_2(t)dt\right]$

$= c_p I_p\left[Dt_s(t_1 - M) - \frac{D(t_1^2 - M^2)}{2} + (1 - x(n))\emptyset_n(t_1 - M) + b(M - t_1)\right] \qquad (3.18)$

On inserting the values of various components from equations $[(3.9) - (3.13)]$, Eq. (3.17) and Eq. (3.18) in the question (3.8) the total profit function for **Case 2,** $z_2(\emptyset_n,\ b)$becomes

$z_2(\emptyset_n,\ b) = p(1 - x(n))\emptyset_n + C_s\,x(n)\emptyset_n - y - \beta\emptyset_n - c_1 b(t_2 + 2t_s) - h_c\left[Dt_1 t_s - \frac{Dt_1^2}{2} + (1 - x(n))\emptyset_n t_1 + b(t_s - t_1)\right] + pI_e\left[\frac{DM^2}{2} + b(M - t_s)\right] + C_s x(n)\emptyset_n I_e(M - t_s) - c_p I_p\left[Dt_s(t_1 - M) - \frac{D(t_1^2 - M^2)}{2} + (1 - x(n))\emptyset_n(t_1 - M) + b(M - t_1)\right] \qquad (3.19)$

**Proposition3.3:** *An order quantity* $\boldsymbol{\phi_n}$ *and credit period back ordered level* $\boldsymbol{b}$ *have an optimal point* $(\phi_n^*, b^*)$,

***Proof:*** As per the conditions of optimality the total profit function has an optimal point $(\boldsymbol{\phi_n^*}, \boldsymbol{b^*})$, if the first order partial derivatives are disappears at point $(\boldsymbol{\phi_n^*}, \boldsymbol{b^*})$, , i.e. $\frac{\partial z_2(\emptyset_n,b)}{\partial\emptyset_n} = 0$ and $\frac{\partial z_2(\emptyset_n,b)}{\partial b} = 0$ Therefore,

$p(1 - x(n)) + C_s\,x(n) - \beta - \frac{2c_1 b}{\lambda} - h_c\Big[\frac{Dt_1}{\lambda} - \frac{D\phi_n t_1'}{\lambda} - Dt_1't_1 + (1 - x(n))\emptyset_n t_1' + (1 - x(n))t_1' + b(\frac{1}{\lambda} - t_1')\Big] - c_p I_p\Big[Dt_1\left(\frac{1}{\lambda} - \frac{t_1'}{2}\right) + Dt_1'\left(\frac{\phi_n}{\lambda} - \frac{t_1}{2}\right) + (1 - x(n))(t_1 - M) + (1 - x(n))\emptyset_n t_1' - DMt_1' + b(\frac{1}{\lambda} - t_1')\Big] = 0 \qquad (3.19a)$

$$\frac{2c_1 b}{D} - \frac{2c_1\emptyset_n}{\lambda} - h_c\left[\frac{b}{D} - \frac{(1 - x(n))\emptyset_n}{D}\right] - c_p I_p\left[\frac{b}{D} - \frac{(1 - x(n))\emptyset_n}{D} + M\right] = 0 \qquad (3.19b)$$

The optimal value of $\boldsymbol{\phi_n}$ **and** $\boldsymbol{b}$ can be obtained from the solution of the above system of two equations.

**Proposition 3.4:** The total cost function $z_2(\emptyset_n,\ b)$ shows jointly concavity for an optimum point $(\boldsymbol{\phi_n}, \boldsymbol{b})$ if $RT - S^2 > 0$, and $R < 0$. i.e

$h_c\Big[\frac{2Dt_1'}{\lambda} - Dt_1'^2 + 2(1 - x(n))t_1' + b(\frac{1}{\lambda} -$

$t_1')\Big] - c_p I_p \Big[2Dt_1'\left(\frac{1}{\lambda} - \frac{t_1'}{2}\right) + 2(1-x(n))t_1'\Big]\Big[\frac{2c_1}{D} + \frac{h_c}{D} + \frac{c_p I_p}{D}\Big] - \Big[\frac{2c_1}{\lambda} - c_p I_p\Big[\frac{(1-x(n))}{D} + t_1'\Big)\Big]\Big]^2 > 0$, and $h_c\Big[\frac{2Dt_1'}{\lambda} - Dt_1'^2 + 2(1-x(n))t_1' + b(\frac{1}{\lambda} - t_1')\Big] c_p I_p \Big[2Dt_1'\left(\frac{1}{\lambda} - \frac{t_1'}{2}\right) + 2(1-x(n))t_1'\Big] < 0.$ (3.19c)

**Proof:** After differentiate twice partially of equation (3.16) with respect to $\emptyset_n$, $\boldsymbol{b}$ can be obtained the derivatives

$$\frac{\partial^2 z_2(\emptyset_n,b)}{\partial \emptyset_n^2} = -h_c\Big[\frac{2Dt_1'}{\lambda} - Dt_1'^2 + 2(1-x(n))t_1' + b(\frac{1}{\lambda} - t_1')\Big] - c_p I_p \Big[2Dt_1'\left(\frac{1}{\lambda} - \frac{t_1'}{2}\right) + 2(1-x(n))t_1'\Big] \quad (3.19d)$$

$$\frac{\partial^2 z_2(\emptyset_n,b)}{\partial b^2} = -\Big[\frac{2c_1}{D} + \frac{h_c}{D} + \frac{c_p I_p}{D}\Big] \quad (3.19e)$$

$$\frac{\partial^2 z_2(\emptyset_n,b)}{\partial \emptyset_n b} = -\frac{2c_1}{\lambda} + c_p I_p\Big[\frac{(1-x(n))}{D} + t_1')\Big] \quad (3.19f)$$

The Hassian matrix H of the profit function $\boldsymbol{z_2}(\boldsymbol{\emptyset_n},\ \boldsymbol{b})$ **is** square matrix of order 2 of partial derivatives of $\boldsymbol{z_2}(\boldsymbol{\emptyset_n},\ \boldsymbol{b})$**:**

$$\mathrm{H}(\boldsymbol{\emptyset_n},\ \boldsymbol{b}) = \begin{bmatrix} \frac{\partial^2 z_2(\emptyset_n,b)}{\partial \emptyset_n^2} & \frac{\partial^2 z_2(\emptyset_n,b)}{\partial \emptyset_n b} \\ \frac{\partial^2 z_2(\emptyset_n,b)}{\partial \emptyset_n b} & \frac{\partial^2 z_2(\emptyset_n,b)}{\partial b^2} \end{bmatrix}$$

The determinant of this Hassian matrix is positive and $\frac{\partial^2 z_2(\emptyset_n,b)}{\partial \emptyset_n b}$ is negative at an optimum point $(\boldsymbol{\phi_n^*}, \boldsymbol{b^*})$, , this condition has been satisfied and illustrated with the numerical data analysis and examples. After simplification of above it can obtain above equation $(3.19c)$, This shows the jointly concavity property of profit function.

### *Case 3: $t_1 \le M \le t_2$*

The situation in which inventory cycle is less then or equal to credit offered period was dealt in this case. Therefore in this situation payable interest by retailer is zero and in addition earned interest for the demand fulfilled for the time period $M - t_1$.

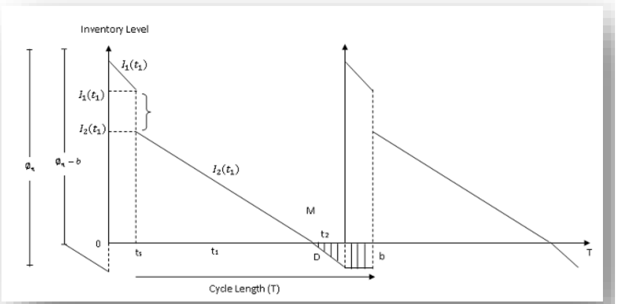


Figure-4

*Interest Earned* $= \mathrm{pI_e}\Big[\int_0^{t_1} Dt\,dt + [Dt_1][M - t_1] + b[M - t_s]\Big] + C_s x(n)\emptyset_n I_e(M - t_s)$

$$= \mathrm{pI_e}\left[\frac{Dt_1^2}{2} + [Dt_1][M - t_1] + b[M - t_s]\right] + C_s x(n)\emptyset_n I_e(M - t_s) \quad (3.20)$$

On inserting the values of various components from equations $[(3.9) - (3.13)]$, Eq. (3.19) and Eq. (3.20) in the question (3.8) the total profit function for **Case 3,** $z_3(\emptyset_n,\ b)$becomes

$$\begin{aligned} z_3(\emptyset_n, b) = {} & p(1 - x(n))\emptyset_n + C_s\,x(n)\emptyset_n - y - \beta \\ & - c_1 b(t_2 + 2t_s) \\ & - h_c\left[Dt_1 t_s - \frac{Dt_1^2}{2} + (1 - x(n))\emptyset_n t_1 + b(t_s - t_1)\right] \\ & + pI_e\left[\frac{Dt_1^2}{2} + [Dt_1][M - t_1] + b[M - t_s]\right] \\ & + C_s x(n)\emptyset_n I_e(M - t_s) \quad (3.21) \end{aligned}$$

**Proposition3.5:** ***An order quantity $\boldsymbol{\phi_n}$ and credit period back ordered level b have an optimal point $(\boldsymbol{\phi_n^*}, \boldsymbol{b^*})$,***

***Proof:*** As per the conditions of optimality the total profit function has an optimal point $(\boldsymbol{\phi_n^*}, \boldsymbol{b^*})$, if the first order partial derivatives are disappears at point $(\boldsymbol{\phi_n^*}, \boldsymbol{b^*})$, i.e. $\frac{\partial z_3(\emptyset_n,b)}{\partial \emptyset_n} = 0$ and $\frac{\partial z_3(\emptyset_n,b)}{\partial b} = 0$ Therefore,

$$p(1 - x(n)) + C_s\,x(n) - \beta - \frac{2c_1 b}{\lambda} - h_c\Big[\frac{Dt_1}{\lambda} - \frac{D\phi_n t_1'}{\lambda} - Dt_1' t_1 + (1 - x(n))\emptyset_n t_1' + (1 - x(n))t_1' + b(\frac{1}{\lambda} - t_1')\Big] - c_p I_p\Big[Dt_1\left(\frac{1}{\lambda} - \frac{t_1'}{2}\right) + Dt_1'\left(\frac{\phi_n}{\lambda} - \frac{t_1}{2}\right) + (1 - x(n))(t_1 - M) + (1 - x(n))\emptyset_n t_1' - DMt_1' + b(\frac{1}{\lambda} - t_1')\Big] = 0 \quad (3.21a)$$

$$\frac{2c_1 b}{D} - \frac{2c_1\emptyset_n}{\lambda} - h_c\left[\frac{b}{D} - \frac{(1 - x(n))\emptyset_n}{D}\right] - c_p I_p\left[\frac{b}{D} - \frac{(1 - x(n))\emptyset_n}{D} + M\right] = 0 \quad (3.21b)$$

The optimal value of $\boldsymbol{\phi_n}$ **and** $\boldsymbol{b}$ can be obtained from the Solution of the above system of two equations.

**Proposition 3.6:** The total cost function $z_3(\emptyset_n,\ b)$ shows jointly concavity for an optimum point $(\boldsymbol{\phi_n}, \boldsymbol{b})$ if $RT - S^2 > 0$, and $R < 0$. i.e

$h_c\Big[\frac{2Dt_1'}{\lambda} - Dt_1'^2 + 2(1-x(n))t_1' + b(\frac{1}{\lambda} - t_1')\Big] - c_p I_p \Big[2Dt_1'\left(\frac{1}{\lambda} - \frac{t_1'}{2}\right) + 2(1-x(n))t_1'\Big]\Big[\frac{2c_1}{D} + \frac{h_c}{D} + \frac{c_p I_p}{D}\Big] - \Big[\frac{2c_1}{\lambda} - c_p I_p\Big[\frac{(1-x(n))}{D} + t_1')\Big]\Big]^2 > 0$, and $h_c\Big[\frac{2Dt_1'}{\lambda} - Dt_1'^2 + 2(1-x(n))t_1' + b(\frac{1}{\lambda} - t_1')\Big] c_p I_p \Big[2Dt_1'\left(\frac{1}{\lambda} - \frac{t_1'}{2}\right) + 2(1-x(n))t_1'\Big] < 0.$ (3.21c)

***Proof:*** *After twicly differentiate partially of equation* (3.21) *with respect to* $\emptyset_n$, $\boldsymbol{b}$ the following derivatives are obtained

$$\frac{\partial^2 z_3(\emptyset_n,b)}{\partial \emptyset_n^2} = -h_c\Big[\frac{2Dt_1'}{\lambda} - Dt_1'^2 + 2(1-x(n))t_1' + b(\frac{1}{\lambda} - t_1')\Big] - c_p I_p \Big[2Dt_1'\left(\frac{1}{\lambda} - \frac{t_1'}{2}\right) + 2(1-x(n))t_1'\Big] \quad (3.21d)$$

$$\frac{\partial^2 z_3(\emptyset_n,b)}{\partial b^2} = -\Big[\frac{2c_1}{D} + \frac{h_c}{D} + \frac{c_p I_p}{D}\Big] \quad (3.21e)$$

$$\frac{\partial^2 z_3\,(\emptyset_n,b)}{\partial \emptyset_n b} = -\frac{2c_1}{\lambda} + c_p I_p\Big[\frac{(1-x(n))}{D} + t_1')\Big] \quad (3.21f)$$

The Hassian matrix H of the profit function $\boldsymbol{z_3(\emptyset_n,\ b)}$ **is** square matrix of order 2 of partial derivatives of $\boldsymbol{z_3(\emptyset_n,\ b)}$**:**

$$H(\boldsymbol{\emptyset_n},\ \boldsymbol{b})=\begin{bmatrix} \frac{\partial^2 z_3(\emptyset_n,b)}{\partial\emptyset_n^2} & \frac{\partial^2 z_3(\emptyset_n,b)}{\partial\emptyset_n b} \\ \frac{\partial^2 z_3(\emptyset_n,b)}{\partial\emptyset_n b} & \frac{\partial^2 z_3(\emptyset_n,b)}{\partial b^2} \end{bmatrix}$$

The determinant of this Hassian matrix is positive and $\frac{\partial^2 z_3(\emptyset_n,b)}{\partial\emptyset_n b}$ is negative at an optimum point $(\boldsymbol{\phi_n^*}, \boldsymbol{b^*})$, this condition has been satisfied and illustrated with the numerical data analysis and examples. After simplification of above it can obtain above equation $(3.19c)$, This shows the jointly concavity property of profit function.
Above equation $(3.21c)$ can be obtained after simplification of above terms. This condition has been satisfied and illustrated numerical analysis and examples.

**Example 1: (Case-1)** An example is developed to illustrate the case 1 of model: The following data set of input paramenters have considered : $D = 76000\ Units/year, y = 100\ per\ cycle\ c_p = \$25\ per\ unit\ p = \$21\ per\ unit\ h_c = \$15\ per\ unit, \beta = \$0.7\ per\ unit, n = 1, M = 0.001\ per\ year,\ I_e = 0.1\ per\ year,$ $\lambda = 60000\ per\ unit\ time, I_e = 0.25\ \ per\ year, c_s = \$10.5\ per\ unit,\quad c_1 = \$8\ per\ year,$ $A = 40, B = 1\ G = 999,\ rt - s^2$ =6.79488E-07

Optimum outputs are: $\phi_n = 14207\ units\ ,\ t_s = 0.237, b = 38.66\ units,\ t_1 = 0.416\ \ Z_1(\phi_n, b) = 311445.$
**Example 2: (case-II)** An example is developed to illustrate the case 1 of model: The following data set of input paramenters have considered: $D = 76000\ Units/year, y = 100\ per\ cycle\ c_p = \$25\ per\ unit\ p = \$21\ per\ unit\ h_c = \$15\ per\ unit, \beta = \$0.7\ per\ unit, n = 1, M = 0.001\ per\ year,\ I_e = 0.1\ per\ year,$ $\lambda = 60000\ per\ unit\ time, I_e = 0.25\ \ per\ year, c_s = \$10.5\ per\ unit,\quad c_1 = \$8\ per\ year,$ $A = 40, B = 1\ G = 999, ,\ rt - s^2$ =2.53452E-07

Optimum outputs are: $\phi_n = 23602\ units\ ,\ t_s = 0.3933,$ $b = 8232\ units,\ t_1 = 0.5832\ Z_2(\phi_n, b) = 438690.$
**Example 3: (case-III)** An example is developed to illustrate the case 1 of model: The following data set of input paramenters are considered: $D = 76000\ Units/year, y = 100\ per\ cycle\ c_p = \$25\ per\ unit\ p = \$21\ per\ unit\ h_c = \$15\ per\ unit, \beta = \$0.7\ per\ unit, n = 1, M = 0.42\ per\ year,$ $I_e = 0.1\ per\ year,\ \lambda = 60000\ per\ unit\ time, I_e = 0.25\ \ per\ year, c_s = \$10.5\ per\ unit,\quad c_1 = \$8\ per\ year,$ $A = 40, B = 1\ G = 999, ,\ rt - s^2$ =*12.30066*.
Optimum outputs are: $\phi_n = 17690\ units\ ,\ t_s = 0.295, b = 7420\ units,\ t_1 = 0.4206\ \ Z_3(\phi_n, b) = 411617.$

## 4. SENSITIVITY ANALYSIS

**Table1: Effects of Learning on Various Outputs For Case 1:**

| n | $\emptyset_n$ | $t_s$ | b | $t_1$ | $(1-x(n))\emptyset_n$ | Profit |
|---|---|---|---|---|---|---|
| 1 | 14207 | 0.23 | 38.65 | 0.42 | 0.96 | 318445 |
| 3 | 14204 | 0.23 | 44.16 | 0.42 | 0.96 | 318547 |
| 5 | 14179 | 0.23 | 79.57 | 0.42 | 0.97 | 319119 |
| 7 | 14092 | 0.23 | 205.81 | 0.42 | 0.98 | 321196 |
| 9 | 14012 | 0.23 | 322.012 | 0.42 | 1.00 | 323114 |

**Table2: Effects of Learning on Various Outputs For Case 2:**

| n | $\emptyset_n$ | $t_s$ | b | $t_1$ | $(1-x(n))\emptyset_n$ | Profit |
|---|---|---|---|---|---|---|
| 1 | 23603 | 0.39 | 8232.566 | 0.58 | 0.96 | 438690 |
| 3 | 23601 | 0.39 | 8237.477 | 0.58 | 0.96 | 438873 |
| 5 | 23581 | 0.39 | 8251.978 | 0.58 | 0.96 | 440093 |
| 7 | 23515 | 0.39 | 8317.42 | 0.58 | 0.98 | 444434 |
| 9 | 23455 | 0.39 | 8380.835 | 0.58 | 0.99 | 448453 |

**Table3: Effects of Learning on Various Outputs For Case 3:**

| n | $\emptyset_n$ | $t_s$ | b | $t_1$ | $(1-x(n))\emptyset_n$ | Profit |
|---|---|---|---|---|---|---|
| 1 | 17690 | 0.29 | 7420 | 0.42 | 0.96 | 411617 |
| 3 | 17688 | 0.29 | 7425 | 0.42 | 0.96 | 411748 |
| 5 | 17666 | 0.29 | 7456 | 0.42 | 0.96 | 412479 |
| 7 | 17594 | 0.29 | 7569 | 0.42 | 0.98 | 415178 |
| 9 | 17528 | 0.29 | 7673 | 0.42 | 0.99 | 417685 |

The comparative study of profit function with respect to decision variable under three difference cases have been prepared and made individual data table for showing effects of learning effects on screening process. Analysis of data table 1 shows that the maximum back ordered level and profit function both are linear positively correlated where as ordered quantity not shows any relation with number of learning efforts see ***Figure 5***.

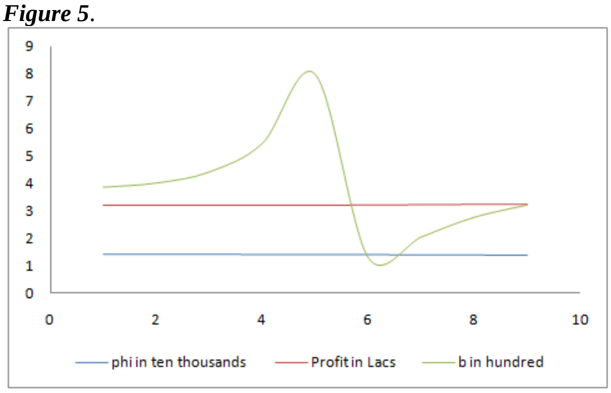


**Figure-5**

Analysis of data table 2 shows that the maximum back ordered level and profit function both are moderate positively correlated where as ordered quantity shows moderate negatively correlated with number of learning efforts see ***Figure 6***.
Analysis of data table 3 shows that the maximum back ordered level and profit function both are strongly positively correlated where as ordered quantity shows moderate negatively correlated with number of learning efforts see ***Figure 7***.

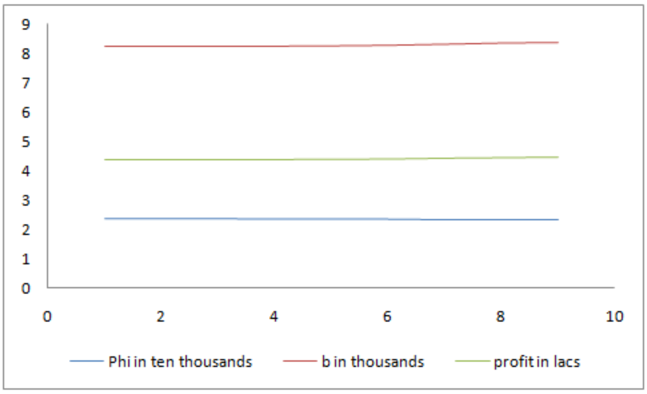


Figure-6

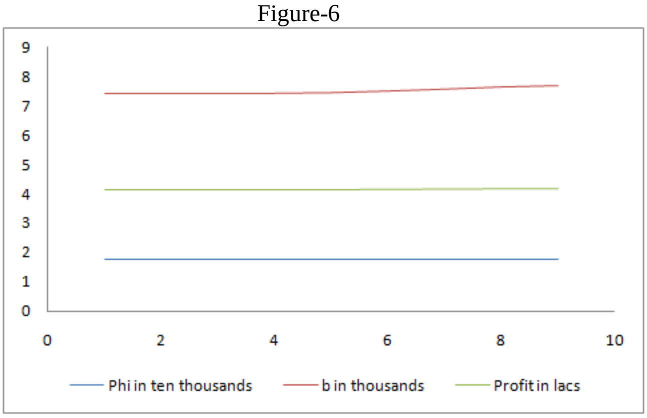


Figure-7

**Table4:Comparative study data table for various cases:**

| Number of learning efforts | Case | $\emptyset_n$ | b | Profit |
|---|---|---|---|---|
| n=1 | 1 | 14207 | 38.65 | 318445 |
| | 2 | 23603 | 8232.57 | 438690(Maximum) In this position case there are no requirement of skilled persons. |
| | 3 | 17690 | 7420.09 | 411617 |
| n=3 | 1 | 14204 | 44.17 | 318547 |
| | 2 | 23601 | 8237.48 | 438873(Maximum) In this position case there is few remarkable improvement from learning skills. |
| | 3 | 17688 | 7425.47 | 411748 |
| n=5 | 1 | 14179 | 79.57 | 319119 |
| | 2 | 23581 | 8251.98 | 440093(Maximum)In this position case there is few remarkable improvement from learning skills. |
| | 3 | 17666 | 7456.30 | 412479 |
| n=7 | 1 | 14092 | 205.81 | 321196 |
| | 2 | 23581 | 8251.98 | 440093(Maximum)In this position case there is some remarkable improvement from learning skills. |
| | 3 | 17594 | 7569.29 | 415178 |
| n=9 | 1 | 14012 | 322.01 | 323114 |
| | 2 | 23455 | 8380.4 | 448453 (Maximum)In this position case there is maximum remarkable improvement from learning skills. |
| | 3 | 17528 | 7673.94 | 417685 |

The behavior of profit function with respect to some another main key parameters as follows:

1. The first partial derivative $\frac{\partial z_1(\emptyset_n,b)}{\partial \lambda}$>0, profit function shows growing property with respect to lambda it mean screening rate increases the profit of vendor
2. The first partial derivative $\frac{\partial z_1(\emptyset_n,b)}{\partial n}$>0 if and only if $p - c_s > 0$, profit function shows growing property with respect to $\lambda$ it means screening rate increases the profit of vendor. where p price of perfect quality item and $c_s$ is a price of imperfect quality items
3. The first partial derivative $\frac{\partial z_1(\emptyset_n,b)}{\partial M}$>0, 0 if and only if $M > t_1$, (it is applicable for case 2 and case 3 only), shows growing property profit with respect to lambda it mean screening rate increases the profit of vendor, where $M$ is a permissible delay period in square off the account.
4. The first partial derivatives $\frac{\partial z_1(\emptyset_n,b)}{\partial c_1}$<0, $\frac{\partial z_1(\emptyset_n,b)}{\partial y}$<0,and $\frac{\partial z_1(\emptyset_n,b)}{\partial h_c}$<0 it means, reducing property with respect to $c_1$, y, $and\ h_c$.

## 5. CONCLUSION

This article suggested an inventory model for a retailer handling imperfect quality items under permissible delay in payments. For separating the perfect and imperfect quality items, a screening process is applied on each batch incorporating with learning effects are also analyzed under allowable shortages and fully backlogged demand. Such strategies have been made in this study so that the order quantity, shortages and the number of repetitions on screening process are optimized and the total profit is maximum as per given limitations. As per sensitivity analysis ***case 2*** $(t_s \leq M \leq t_1)$, is the most favorable case for available circumstances.

## Suggestions and Recommendations:

(i) It is a suggestion for retailer to make such strategy with their supplier for maintaining the limitations of case2.

(ii) Retailer's must recruit skilled and efficient those workers whose engaged in screening process.

(iii) Retailer can be also applied rework process on imperfect quality items to make extra sales revenue and consequently earn extra interest and profit.

Extensions:

(i) This article can be extended by incorporating credit period to their customers.

(ii) This article is made for fully backlogged demand, it can be extended by incorporating partial backlogged demand.

(iii) This article can be also extended by incorporating rework process on imperfect quality items.

## 6. ACKNOWLEDGMENTS

Our thanks to the experts who have contributed towards development of the template.

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