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Mean Harmonic Energy of a Graph

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ABSTRACT

In this paper, the concept of mean harmonic energy of a graph denoted by $E_{mh}(G)$ is introduced, the mean harmonic energy of some classes of graphs is computed. Also some basic properties of mean harmonic energy and some bounds for mean harmonic energy of a graph have been obtained.

General Terms

AMS, Subject Classification 05C50, 05C99.

Keywords

Eigenvalue of a graph, Energy, Mean Harmonic Energy

1. INTRODUCTION

Throughout this paper, we consider finite, simple, and undirected graphs. Let G be a graph of order n and size m whose vertex and edge sets are $V(G)$ and $E(G)$, respectively. The energy $E(G)$ of a graph G, defined as the sum of the absolute values of its eigenvalues, belongs to the most popular graph invariants in chemical graph theory. It originates from the π-electron energy in the Huckel molecular orbital model, but has also gained purely mathematical interest. This concept was introduced in the subject of chemistry by I. Jutman and is intensively studied since it can be used to approximate the total π-electron of a molecule [6].

Spectral graph theory [3] is an attractive research area that finds the relation between the combinatorial properties of graphs and the algebraic properties of associated matrices, as well as applications of those connections. More broadly, it searches for the link between the discrete universe and the continuous one by employing geometric, analytic, and algebraic techniques. For a graph G, the conventional adjacency matrix, denoted as $A(G)$, is one of the effective tools in this domain. The (i, j)- element of $A(G)$ is 1 when v_i is adjacent to v_j and 0 otherwise. Its characteristic polynomial is $\psi_A(G, \lambda) = det(\lambda I_n - A(G))$, where I_n is the identity matrix. Since $A(G)$ is real and symmetric, one can arrange its eigenvalues as $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$. The set of eigenvalues of the graph with their multiplicities is known as spectrum of the graph. The energy of the graph G is defined as the sum of the absolute values of its eigenvalues [6, 11]:

$$E(G) = \sum_{i=1}^{n} |\lambda_i|.$$

This concept was introduced almost 46 years ago [6] and has been extensively investigated [2, 7, 8]. Eventually, numerous other graph energies have been invented, based on eigenvalues of matrices different from the adjacency matrix; for more details see [1, 9, 12, 13]. Motivated by the works [10, 15] and the large applications of these energies, here, we introduce the mean harmonic energy.

We will now describe some terms and symbols that will be utilized all through the paper. The degree of a vertex v, denoted by $d(i)$, is the number of edges of G incident with v. We denote by $K_n, K_{n,m}, K_{1,n}$, S_n^0 and F_n the complete graph, complete bipartite graph, star, crown and friendship, respectively.
In this paper, we introduce the concept of mean harmonic matrix **MH**.

DEFINITION 1. *Let $G = (V, E)$ be a simple connected graph with n vertices $v_1, v_2, ..., v_n$, and let d_i be the degree of $v_i, i = 1, 2, ..., n$. Then the mean harmonic matrix **MH=MH(G)** of G is the $n \times n$ matrix, whose (i, j)-entry is as follows:*

$$a_{ij} = \begin{cases} \frac{2}{\frac{1}{d_i} + \frac{1}{d_j}}, & \text{if } v_i v_j \in E(G); \\ 0, & \text{otherwise.} \end{cases}$$

The characteristic polynomial of $MH(G)$ denoted by $f_n(G, \lambda)$, is defined as

$$f_n(G, \lambda) := det\left(\lambda I - MH(G)\right).$$

The eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$ of **MH**(G) form the mean harmonic spectrum or the **MH**- spectrum of G. **MH** is a real symmetric matrix. Therefore, its eigenvalues are real numbers, and $\sum_{i=1}^{n} \lambda_i = 0$.

2. MEAN HARMONIC ENERGY

DEFINITION 2. *The mean harmonic energy $E_{mh}(G)$ of a graph G is*

$$E_{mh}(G) = \sum_{i=1}^{n} |\lambda_i|.$$

EXAMPLE 1. *Let G be a graph in Figure 1.*

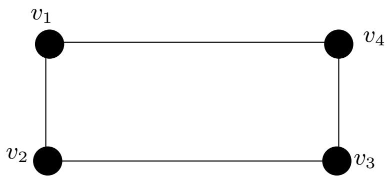

Figure 1: C_4

Then

$$MH(G) = \begin{pmatrix} 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 \end{pmatrix}$$

The characteristic polynomial of $MH(G)$ is $f_n(G,\lambda) = \lambda^4 - 16\lambda^2$, the mean harmonic eigenvalues are $\lambda_1 = 0,\ \lambda_2 = 0,\ \lambda_3 = -4,\ \lambda_4 = 4$. Therefore the mean harmonic energy of G is

$$E_{mh}(G) = 8.$$

We need the following to prove main results.

THEOREM 3. *[14] Let a_i and b_i be nonnegative real numbers, then*

$$\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - (\sum_{i=1}^{n} a_i b_i)^2 \leq \frac{n^2}{4}(M_1 M_2 - m_{1m_2})^2. \quad (1)$$

Where, $M_1 = max(a_i), M_2 = max(b_i), m_1 = min(a_i), m_2 = min(b_i)$, also $i = 1, 2, ..., n$.

2.1 Some basic properties of mean harmonic energy

In this section, some properties of characteristic polynomials of mean harmonic matrix of a graph G have been introduced.

THEOREM 4. *Let G be a graph of order n, size m. Let $f_n(G,\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + ... + a_n$ be the characteristic polynomial of mean harmonic matrix of a graph G. Then*

(1) $a_0 = 1$.
(2) $a_1 = 0$
(3) $a_2 = -q$ *such that* $q = \sum_{i<j} \left(\frac{2}{\frac{1}{d_i} + \frac{1}{d_j}} \right)^2$
(4) $a_n = det(MH(G))$

PROOF. (1) Clearly, $a_0 = 1$ from the definition of $f_n(G,\lambda)$.

(2) Since diagonal elements of $MH(G)$ is equal to 0, so $a_1 = trace(MH(G)) = 0$.

(3) $(-1)^2 a_2$ is equal to the sum of determinants of all 2×2 principal sub matrices of $MH(G)$, that is

$$\begin{aligned} a_2 &= \sum_{1 \leq i < j \leq n} \begin{vmatrix} c_{ii} & c_{ij} \\ c_{ji} & c_{jj} \end{vmatrix} \\ &= \sum_{1 \leq i < j \leq n} (c_{ii}c_{jj} - c_{ij}c_{ji}) \\ &= \sum_{1 \leq i < j \leq n} c_{ii}c_{jj} - \sum_{1 \leq i < j \leq n} c_{ij}^2 \\ &= -\sum_{i<j} \left(\frac{2}{\frac{1}{d_i} + \frac{1}{d_j}} \right)^2. \end{aligned}$$

(4) It is clear that $a_n = det(MH(G))$.

□

THEOREM 5. *Let G be a graph of order n. Let $\lambda_1, \lambda_2, ..., \lambda_n$ be the eigenvalues of $MH(G)$. Then*

(i) $\sum_{i=1}^{n} \lambda_i = 0.$

(ii) $\sum_{i=1}^{n} \lambda_i^2 = 2q,$ *where* $q = \sum_{i<j} \left(\frac{2}{\frac{1}{d_i} + \frac{1}{d_j}} \right)^2.$

PROOF. (i) Since the sum of eigenvalues of $MH(G)$ is the trace of $MH(G)$, then

$$\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} a_{ii} = 0.$$

(ii) We know that

$$\begin{aligned} \sum_{i=1}^{n} \lambda_i^2 &= \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}a_{ji} \\ &= \sum_{i=1}^{n} a_{ii}^2 + \sum_{i \neq j}^{n} a_{ij}a_{ji} \\ &= \sum_{i=1}^{n} a_{ii}^2 + 2\sum_{i<j}^{n} a_{ij}^2 \\ &= 0 + 2\sum_{i<j} \left(\frac{2}{\frac{1}{d_i} + \frac{1}{d_j}} \right)^2 \\ &= 2q. \end{aligned}$$

□

LEMMA 6. *Let G be a connected (n,m) graph. Then $traceMH^2(G) \leq traceMH^2(K_n) = n(n-1)^3$, and the bound is sharp for $G \cong K_n$.*

PROOF. Since

$$MH(K_n)_{ij} = \begin{cases} n-1, & \text{if } v_iv_j \in E\,; \\ 0, & \text{Otherwise.} \end{cases}$$

We have, for $i \neq j$

$$MH^2(K_n)_{ij} = (n-2)(n-1)^2,$$

and for $i = j$,

$$MH^2(K_n)_{ii} = (n-1)(n-1)^2,$$

implying that

$$traceMH^2(G) = n(n-1)^3.$$

□

3. MEAN HARMONIC ENERGY OF SOME STANDARD GRAPHS

In this section, the exact values of the mean harmonic energy of some standard graphs was investigated.

THEOREM 7. *For the complete graph $K_n,\ n \geq 2,$*

$$E_{mh}(K_n) = 2(n-1)^2.$$

PROOF. Let K_n be the complete graph with vertex set $V = \{v_1, v_2, \cdots, v_n\}$. Then

$$MH(K_n) = \begin{pmatrix} 0 & n-1 & n-1 & \cdots & n-1 & n-1 & \cdots & n-1 & n-1 \\ n-1 & 0 & n-1 & \cdots & n-1 & n-1 & \cdots & n-1 & n-1 \\ n-1 & n-1 & 0 & \cdots & n-1 & n-1 & \cdots & n-1 & n-1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ n-1 & n-1 & n-1 & \cdots & 0 & n-1 & \cdots & n-1 & n-1 \\ n-1 & n-1 & n-1 & \cdots & n-1 & 0 & \cdots & n-1 & n-1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ n-1 & n-1 & n-1 & \cdots & n-1 & n-1 & \cdots & 0 & n-1 \\ n-1 & n-1 & n-1 & \cdots & n-1 & n-1 & \cdots & n-1 & 0 \end{pmatrix}_{n\times n}$$

The respective characteristic polynomial is

$$f_n(K_n, \lambda) = \begin{vmatrix} \lambda & -(n-1) & -(n-1) & \cdots & -(n-1) & -(n-1) & \cdots & -(n-1) & -(n-1) \\ -(n-1) & \lambda & -(n-1) & \cdots & -(n-1) & -(n-1) & \cdots & -(n-1) & -(n-1) \\ -(n-1) & -(n-1) & \lambda & \cdots & -(n-1) & -(n-1) & \cdots & -(n-1) & -(n-1) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -(n-1) & -(n-1) & -(n-1) & \cdots & \lambda & -(n-1) & \cdots & -(n-1) & -(n-1) \\ -(n-1) & -(n-1) & -(n-1) & \cdots & -(n-1) & \lambda & \cdots & -(n-1) & -(n-1) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -(n-1) & -(n-1) & -(n-1) & \cdots & -(n-1) & -(n-1) & \cdots & \lambda & -(n-1) \\ -(n-1) & -(n-1) & -(n-1) & \cdots & -(n-1) & -(n-1) & \cdots & -(n-1) & \lambda \end{vmatrix}$$

$= (\lambda - (n-1)^2)(\lambda + (n-1))^{n-1}$.
Hence, mean harmonic eigenvalues are $\lambda = (n-1)^2\ [one\,time],\ \lambda = -(n-1)\ [n-1\,times]$.
Therefore, the mean harmonic energy of a complete graph is

$$E_{mh}(K_n) = 2(n-1)^2.$$

□

THEOREM 8. *For $n \geq 2$, the mean harmonic energy of a star graph $K_{1,n-1}$ is equal to $\frac{4}{n}\sqrt{(n-1)^3}$.*

PROOF. Let $K_{1,n-1}$ be a star graph with vertex set $V = \{v, v_1, v_2, \cdots, v_{n-1}\}$, v is the center. Then

$$MH(K_{1,n-1}) = \begin{pmatrix} 0 & \frac{2}{n}(n-1) & \frac{2}{n}(n-1) & \frac{2}{n}(n-1) & \cdots & \frac{2}{n}(n-1) \\ \frac{2}{n}(n-1) & 0 & 0 & 0 & \cdots & 0 \\ \frac{2}{n}(n-1) & 0 & 0 & 0 & \cdots & 0 \\ \frac{2}{n}(n-1) & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ \frac{2}{n}(n-1) & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{n\times n}$$

The characteristic polynomial of $MH(K_{1,n-1})$ is

$$f_n(K_{1,n-1}, \lambda) = \begin{vmatrix} \lambda & -\frac{2}{n}(n-1) & -\frac{2}{n}(n-1) & -\frac{2}{n}(n-1) & \cdots & -\frac{2}{n}(n-1) \\ -\frac{2}{n}(n-1) & \lambda & 0 & 0 & \cdots & 0 \\ -\frac{2}{n}(n-1) & 0 & \lambda & 0 & \cdots & 0 \\ -\frac{2}{n}(n-1) & 0 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{2}{n}(n-1) & 0 & 0 & 0 & \cdots & \lambda \end{vmatrix}$$

$$= (\lambda)^{n-1}(\lambda - \tfrac{2}{n}\sqrt{(n-1)^3})(\lambda + \tfrac{2}{n}\sqrt{(n-1)^3}).$$

Hence, mean harmonic eigenvalues are $\lambda = 0\ [n-1\ times],\ \lambda = \frac{2}{n}\sqrt{(n-1)^3}\ [one\ \ time]$, and $\lambda = -\frac{2}{n}\sqrt{(n-1)^3}\ [one\ \ time]$.
Therefore, the mean harmonic energy of a star graph is

$$E_{mh}(K_{1,n-1}) = \frac{4}{n}\sqrt{(n-1)^3}.$$

□

THEOREM 9. *For the complete bipartite graph $K_{n,n},\ n \geq 2$, the mean harmonic energy is $2n^2$.*

PROOF. For the complete bipartite graph $K_{n,n},\ n \geq 2$ with vertex set $V = \{u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n\}$. Then

$$MH(K_{n,n}) = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & n & n & n & \cdots & n \\ 0 & 0 & 0 & \cdots & 0 & n & n & n & \cdots & n \\ 0 & 0 & 0 & \cdots & 0 & n & n & n & \cdots & n \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & n & n & n & \cdots & n \\ n & n & n & \cdots & n & 0 & 0 & 0 & \cdots & 0 \\ n & n & n & \cdots & n & 0 & 0 & 0 & \cdots & 0 \\ n & n & n & \cdots & n & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ n & n & n & \cdots & n & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{(2n)\times(2n)}$$

The characteristic polynomial of $MH(K_{n,n})$,

$$f_{2n}(K_{n,n}, \lambda) = \begin{vmatrix} \lambda & 0 & 0 & \cdots & 0 & -n & -n & -n & \cdots & -n \\ 0 & \lambda & 0 & \cdots & 0 & -n & -n & -n & \cdots & -n \\ 0 & 0 & \lambda & \cdots & 0 & -n & -n & -n & \cdots & -n \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & -n & -n & -n & \cdots & -n \\ -n & -n & -n & \cdots & -n & \lambda & 0 & 0 & \cdots & 0 \\ -n & -n & -n & \cdots & -n & 0 & \lambda & 0 & \cdots & 0 \\ -n & -n & -n & \cdots & -n & 0 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -n & -n & -n & \cdots & -n & 0 & 0 & 0 & \cdots & \lambda \end{vmatrix}$$

$$= (\lambda)^{2n-2}\left[\lambda - n^2\right]\left[\lambda + n^2\right].$$

Hence, mean harmonic eigenvalues are $\lambda = 0\ \ [2n-2\ \ times],\ \lambda = n^2\ [one\ \ time],\ \lambda = -n^2\ [one\ \ time]$.
Therefore, the mean harmonic energy of a complete bipartite graph is $E_{mh}(K_{n,n}) = 2n^2$. □

DEFINITION 10. *[4] The crown graph S_n^0 for an integer $n \geq 2$ is the graph with vertex set $\{u_1, u_2, ..., u_n, v_1, v_2, ..., v_n\}$ and edge set $\{u_i u_j : 1 \leq i, j \leq n, i \neq j\}$. S_n^0 coincides with the complete bipartite graph $K_{n,n}$ with horizontal edges removed.*

THEOREM 11. *For $n > 2$, the mean harmonic energy of the crown graph S_n^0 is equal to*

$$E_{mh}(S_n^0) = 4(n-1)^2.$$

PROOF. Let S_n^0 be the crown graph, having vertex set $V(S_n^0) = \{u_1, u_2, ..., u_n, v_1, v_2, ..., v_n\}$. Then

$$MH(S_n^0) = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & n-1 & n-1 & n-1 \\ 0 & 0 & 0 & 0 & \cdots & n-1 & 0 & n-1 & n-1 \\ 0 & 0 & 0 & 0 & \cdots & n-1 & n-1 & 0 & n-1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & n-1 & n-1 & n-1 & 0 \\ 0 & n-1 & n-1 & n-1 & \cdots & 0 & 0 & 0 & 0 \\ n-1 & 0 & n-1 & n-1 & \cdots & 0 & 0 & 0 & 0 \\ n-1 & n-1 & 0 & n-1 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ n-1 & n-1 & n-1 & 0 & \cdots & 0 & 0 & 0 & 0 \end{pmatrix}_{2n\times 2n}$$

The characteristic polynomial of $MH(S_n^0)$ is

$$f_{2n}(S_n^0, \lambda) = \begin{vmatrix} \lambda & 0 & 0 & 0 & \cdots & 0 & 1-n & 1-n & 1-n \\ 0 & \lambda & 0 & 0 & \cdots & 1-n & 0 & 1-n & 1-n \\ 0 & 0 & \lambda & 0 & \cdots & 1-n & 1-n & 0 & 1-n \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \lambda & \cdots & 1-n & 1-n & 1-n & 0 \\ 0 & 1-n & 1-n & 1-n & \cdots & \lambda & 0 & 0 & 0 \\ 1-n & 0 & 1-n & 1-n & \cdots & 0 & \lambda & 0 & 0 \\ 1-n & 1-n & 0 & 1-n & \cdots & 0 & 0 & \lambda & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 1-n & 1-n & 1-n & 0 & \cdots & 0 & 0 & 0 & \lambda \end{vmatrix}$$

$$= (\lambda - (n-1)^2)(\lambda + (n-1)^2)\,[\lambda - (n-1)]^{n-1}\,[\lambda + (n-1$$

Therefore, mean harmonic eigenvalues are $\lambda = (n-1)^2\ [one\,time],\ \lambda = -(n-1)^2\ [one\,time],\ \lambda = n-1\ [n-1\,times],\ \lambda = -(n-1)\ [n-1\,times]$.

Hence,

$$E_{mh}(S_n^0) = 4(n-1)^2.$$

□

DEFINITION 12. *[5] A Friendship graph F_n is a one point union of n copies of cycle C_3. $V(F_n) = 2n + 1$.*

THEOREM 13. *For the A Friendship graph F_n, $n \geq 2$, the mean harmonic energy is*

$$E_{mh}(F_n) = 4n + 2\sqrt{\frac{(n+1)^2 + 2^5 n^3}{(n+1)^2}}.$$

PROOF. Let F_n, $n \geq 2$ be the Friendship graph, having vertex set $V(F_n) = \{v_0, v_1, ..., v_{2n}\}$, v_0 is the center. Then

$$MH(F_n) = \begin{pmatrix} 0 & \frac{4n}{n+1} & \frac{4n}{n+1} & \frac{4n}{n+1} & \frac{4n}{n+1} & \cdots & \frac{4n}{n+1} & \frac{4n}{n+1} \\ \frac{4n}{n+1} & 0 & 2 & 0 & 0 & \cdots & 0 & 0 \\ \frac{4n}{n+1} & 2 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \frac{4n}{n+1} & 0 & 0 & 0 & 2 & \cdots & 0 & 0 \\ \frac{4n}{n+1} & 0 & 0 & 2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{4n}{n+1} & 0 & 0 & 0 & 0 & \cdots & 0 & 2 \\ \frac{4n}{n+1} & 0 & 0 & 0 & 0 & \cdots & 2 & 0 \end{pmatrix}_{(2n+1)\times(2n+1)}$$

The characteristic polynomial of $MH(F_n)$ is

$$f_{2n+1}(F_n, \lambda) = \begin{vmatrix} \lambda & -\frac{4n}{n+1} & -\frac{4n}{n+1} & -\frac{4n}{n+1} & -\frac{4n}{n+1} & \cdots & -\frac{4n}{n+1} & -\frac{4n}{n+1} \\ -\frac{4n}{n+1} & \lambda & -2 & 0 & 0 & \cdots & 0 & 0 \\ -\frac{4n}{n+1} & -2 & \lambda & 0 & 0 & \cdots & 0 & 0 \\ -\frac{4n}{n+1} & 0 & 0 & \lambda & -2 & \cdots & 0 & 0 \\ -\frac{4n}{n+1} & 0 & 0 & -2 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -\frac{4n}{n+1} & 0 & 0 & 0 & 0 & \cdots & \lambda & -2 \\ -\frac{4n}{n+1} & 0 & 0 & 0 & 0 & \cdots & -2 & \lambda \end{vmatrix}$$

$$= (\lambda-2)^{n-1}(\lambda+2)^n \left[\lambda^2 - 2\lambda - \tfrac{2^5 n^3}{(n+1)^2}\right].$$

Hence, mean harmonic eigenvalues are $\lambda = 2\ [n-1\ times]$, $\lambda = -2\ [n\ times]$, $\lambda = 1 \pm \sqrt{\frac{(n+1)^2+2^5n^3}{(n+1)^2}}\ [one\ time\ each]$. Therefore, the mean harmonic energy of a friendship graph is

$$E_{mh}(F_n) = 4n + 2\sqrt{\frac{(n+1)^2 + 2^5 n^3}{(n+1)^2}}.$$

□

4. BOUNDS FOR MEAN HARMONIC ENERGY OF A GRAPH

In this section, some bounds for mean harmonic energy of graphs have been studied.

THEOREM 14. *Let G be a graph of order n and size m. Then*

$$\sqrt{2\sum_{i<j}\left(\frac{2}{\frac{1}{d_i}+\frac{1}{d_j}}\right)^2} \leq E_{mh}(G) \leq \sqrt{2n\sum_{i<j}\left(\frac{2}{\frac{1}{d_i}+\frac{1}{d_j}}\right)^2}.$$

PROOF. Let $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_n$ be the eigenvalues of $MH(G)$. By the Cauchy-Schwartz inequality, we have

$$\left(\sum_{i=1}^n a_i b_i\right)^2 \leq \left(\sum_{i=1}^n a_i^2\right)\left(\sum_{i=1}^n b_i^2\right).$$

Let $a_i = 1$ and $b_i = |\lambda_i|$. Then

$$(E_{mh}(G))^2 = \left(\sum_{i=1}^n |\lambda_i|\right)^2 \leq \left(\sum_{i=1}^n 1\right)\left(\sum_{i=1}^n \lambda_i^2\right) \leq 2n\sum_{i<j}\left(\frac{2}{\frac{1}{d_i}+\frac{1}{d_j}}\right.$$

Therefore, the upper bound holds.
Now, since

$$\left(\sum_{i=1}^n |\lambda_i|\right)^2 \geq \sum_{i=1}^n \lambda_i^2,$$

we have

$$(E_{mh}(G))^2 \geq \sum_{i=1}^n \lambda_i^2 = 2\sum_{i<j}\left(\frac{2}{\frac{1}{d_i}+\frac{1}{d_j}}\right)^2.$$

Therefore.

$$E_{mh}(G) \geq \sqrt{2\sum_{i<j}\left(\frac{2}{\frac{1}{d_i}+\frac{1}{d_j}}\right)^2}.$$

□

THEOREM 15. *Let G be a connected graph of order and size n and m, respectively. If $\Delta = det(MH(G))$, then*

$$E_{mh}(G) \geq \sqrt{2\sum_{i<j}\left(\frac{2}{\frac{1}{d_i}+\frac{1}{d_j}}\right)^2 + n(n-1)\Delta^{2/n}}.$$

PROOF. Since

$$(E_{mh}(G))^2 = \left(\sum_{i=1}^n |\lambda_i|\right)^2 = \left(\sum_{i=1}^p |\lambda_i|\right)\left(\sum_{i=1}^n |\lambda_i|\right) = \sum_{i=1}^n |\lambda_i|^2 + \sum_{i\neq j} |\lambda_i||\lambda_j|.$$

using the inequality between the arithmetic and geometric means, we have

$$\frac{1}{n(n-1)}\sum_{i\neq j}|\lambda_i||\lambda_j| \geq \left(\prod_{i\neq j}|\lambda_i||\lambda_j|\right)^{1/[n(n-1)]}.$$

Then

$$\begin{aligned}(E_{mh}(G))^2 &\geq \sum_{i=1}^n |\lambda_i|^2 + n(n-1)\left(\prod_{i\neq j}|\lambda_i||\lambda_j|\right)^{1/[n(n-1)]} \\ &\geq \sum_{i=1}^n |\lambda_i|^2 + n(n-1)\left(\prod_{i=j}|\lambda_i|^{2(n-1)}\right)^{1/[n(n-1)]} \\ &= \sum_{i=1}^n |\lambda_i|^2 + n(n-1)\left|\prod_{i=j}\lambda_i\right|^{2/n}\end{aligned}$$

$$= 2\sum_{i<j}\left(\frac{2}{\frac{1}{d_i}+\frac{1}{d_j}}\right)^2 + n(n-1)\Delta^{2/n}.$$

□

THEOREM 16. *If G is a graph of order n, then for any mean harmonic eigenvalue λ_j, we have*

$$|\lambda_j| \leq (n-1)^2.$$

PROOF. By Lemma 2.5, $trace(MH^2(K_n)) = n(n-1)^3$. Hence for every graph G of order n with mean harmonic eigenvalues $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_n$, we have

$$\sum_{i=1}^{n} |\lambda_i^2| \leq n(n-1)^3.$$

By the Cauchy-Schwarz inequality,

$$(\sum_{i=1, i\neq j}^{n} \lambda_i)^2 = (n-1)\sum_{i=, i\neq j}^{n} \lambda_i^2.$$

Since $\sum_{i=1}^{n} \lambda_i = 0$ and $\sum_{i=1}^{n} \lambda_i^2 = 2\sum_{i<j}\left(\frac{2}{\frac{1}{d_i}+\frac{1}{d_j}}\right)^2$, we get

$$\lambda_j^2 \leq (n-1)\left(n(n-1)^3 - \lambda_j^2\right),$$

which implies that

$$|\lambda_j| \leq (n-1)^2.$$

□

THEOREM 17. *Let G be a graph of order n and size m. Then*

$$E_{mh}(G) \geq \sqrt{2n\sum_{i<j}\left(\frac{2}{\frac{1}{d_i}+\frac{1}{d_j}}\right)^2 - \frac{n^2}{4}(|\lambda_1| - |\lambda_n|)^2}.$$

PROOF. Let $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_n$ be the mean harmonic eigenvalues of G. Substituting $a_i = 1$ and $b_i = |\lambda_i|$ in the equation (1), we get

$$\sum_{i=1}^{n} 1^2 \sum_{i=1}^{n} |\lambda_i^2| - (\sum_{i=1}^{n} |\lambda_i^2|)^2 \leq \tfrac{n^2}{4}(|\lambda_1| - |\lambda_n|)^2$$

$$2n\sum_{i<j}\left(\frac{2}{\frac{1}{d_i}+\frac{1}{d_j}}\right)^2 - (E_{mh}(G))^2 \geq \tfrac{n^2}{4}(|\lambda_1| - |\lambda_n|)^2.$$

Then,

$$E_{mh}(G) \geq \sqrt{2n\sum_{i<j}\left(\frac{2}{\frac{1}{d_i}+\frac{1}{d_j}}\right)^2 - \frac{n^2}{4}(|\lambda_1| - |\lambda_n|)^2}.$$

□

5. CONCLUSION

In this paper, we obtain the bounds for mean harmonic energy of graphs and present its exact value for complete graph, complete bipartite graph, star graph, crown graph and a friendship graph. The mean harmonic energy of several other families of graphs is an open problem.

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